

REAL HOMOGENOUS SPACES, GALOIS COHOMOLOGY, AND REEDER PUZZLES

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ABSTRACT. Let G be a simply connected absolutely simple algebraic group defined over the field of real numbers \mathbb{R} . Let H be a simply connected semisimple \mathbb{R} -subgroup of G . We consider the homogeneous space $X = G/H$. We ask: How many connected components has $X(\mathbb{R})$?

In the paper we give a method of answering this question. Our method is based on our solutions of generalized Reeder puzzles.

0. INTRODUCTION

In this paper by a semisimple or reductive group we always mean a *connected* semisimple or reductive group, resp. Let G be a *simply connected* absolutely simple algebraic group over the field of real numbers \mathbb{R} . Let $H \subset G$ be a *simply connected* semisimple \mathbb{R} -subgroup. We consider the homogeneous space $X = G/H$, which is an algebraic variety over \mathbb{R} . The topological space $X(\mathbb{R})$ of \mathbb{R} -points of X need not be connected. We ask

Question 0.1. *How many connected components has $X(\mathbb{R})$?*

The group of \mathbb{R} -points $G(\mathbb{R})$ acts on the left on $X(\mathbb{R})$, and we consider the orbits of this action. By Lemma 15.1 below, the set of connected components of $X(\mathbb{R})$ is the set of orbits $G(\mathbb{R}) \backslash X(\mathbb{R})$ of $G(\mathbb{R})$ in $X(\mathbb{R})$. On the other hand, there is a canonical bijection

$$(1) \quad G(\mathbb{R}) \backslash X(\mathbb{R}) \xrightarrow{\sim} \ker [H^1(\mathbb{R}, H) \rightarrow H^1(\mathbb{R}, G)],$$

see Serre [Se94, Section I.5.4, Corollary 1 of Proposition 35], where $H^1(\mathbb{R}, G)$ denotes the first (nonabelian) Galois cohomology of G . We see that Question 0.1 is equivalent to the following question:

Question 0.2. *What is the cardinality of the finite set $\ker [H^1(\mathbb{R}, H) \rightarrow H^1(\mathbb{R}, G)]$?*

In this paper we give a method of answering Question 0.2 (and hence, Question 0.1) using our solutions of generalized Reeder puzzles.

Let G be a *simply connected*, absolutely simple, simply-laced, compact \mathbb{R} -group. Let $T \subset G$ be a maximal torus. Let Π be a basis of the root system $R(G_{\mathbb{C}}, T_{\mathbb{C}})$. Let $D = D(G_{\mathbb{C}}, T_{\mathbb{C}}, \Pi)$ be the Dynkin diagram of G with the set of vertices numbered by $1, 2, \dots, n$. By a *labeling* of D we mean a family $\mathbf{a} = (a_i)_{i=1, \dots, n}$, where $a_i \in \mathbb{Z}/2\mathbb{Z}$. In other words, at any vertex i we write a label $a_i = 0, 1$. We consider the set $L(D)$ of the labelings of D , it is an n -dimensional vector space over the field $\mathbb{Z}/2\mathbb{Z}$.

For any $i \in \Pi$ we define the *move* \mathcal{M}_i applied to a labeling \mathbf{a} : if the vertex i has an *odd* number of neighbors with 1, \mathcal{M}_i *changes* a_i (from 0 to 1 or from 1 to 0), otherwise it does nothing. Clearly $\mathcal{M}_i(\mathcal{M}_i(\mathbf{a})) = \mathbf{a}$. We say that two labelings \mathbf{a}, \mathbf{a}' are *equivalent* if we can pass from \mathbf{a} to \mathbf{a}' by a finite sequence of moves. This is indeed an equivalence relation on $L(D)$. We denote the corresponding set of equivalence classes by $\text{Cl}(D)$. It is the set of

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orbits of the Weyl group W acting on $L(D)$, and we denote it also by $\text{Orb}(D)$ in Sections 6–14 below. The set $\text{Cl}(D)$ has a neutral element $[0]$, the class of the zero labeling 0. To solve the puzzle means to describe the set of equivalence classes $\text{Cl}(D)$ and to describe each equivalence class.

This is original Reeder’s puzzle [R05], except that Reeder formulated his puzzle for any simply-laced graph, not necessarily a simply-laced Dynkin diagram. For a compact, simply connected, simply-laced group G with Dynkin diagram D , the pointed set $\text{Cl}(D)$ is in a bijection with $H^1(\mathbb{R}, G)$. In order to deal with non-simply-laced and noncompact groups, we generalize the puzzle.

We permit non-simply-laced Dynkin diagrams. Then, when counting the number of neighbors with 1 of a given vertex i , we do not count the *shorter* neighbors of i connected with i by a *double* edge. In other words, “the long roots don’t see the short roots”.

We consider also colored Dynkin diagrams, which correspond to non-compact inner forms of compact groups. A *coloring* of a Dynkin diagram D is a family

$$\mathbf{t} = (t_i)_{i=1, \dots, n}, \quad t_i \in \mathbb{Z}/2\mathbb{Z}.$$

If $t_i = 1$, we color vertex i in black, otherwise we leave it white. When vertex i is white, the move \mathcal{M}_i acts as above. When i is black, the move \mathcal{M}_i changes a_i if i has an *even* number of neighbors with 1, and does nothing otherwise. We write sometimes $L(D, \mathbf{t})$ for the set $L(D)$ with this Reeder puzzle. We denote the corresponding set of equivalence classes by $\text{Cl}(D, \mathbf{t})$. If $\mathbf{t} = \mathbf{0} = (0, \dots, 0)$, we have $\text{Cl}(D, \mathbf{0}) = \text{Cl}(D)$. Note that if D has a *black* vertex i , then the move \mathcal{M}_i takes the zero labeling 0 to a nonzero labeling and hence does not respect the group structure in $L(D)$.

We recall the definition of $H^1(\mathbb{R}, G)$. Let G be a linear algebraic group over \mathbb{R} . We denote by $G(\mathbb{C})$ the set of \mathbb{C} -points of G . The first Galois cohomology set $H^1(\mathbb{R}, G)$ is, by definition, $Z^1(\mathbb{R}, G)/\sim$, where the set of 1-cocycles $Z^1(\mathbb{R}, G)$ is defined by $Z^1(\mathbb{R}, G) = \{z \in G(\mathbb{C}) \mid z\bar{z} = 1\}$, and two 1-cocycles $z, z' \in Z^1(\mathbb{R}, G)$ are cohomologous (we write $z \sim z'$) if $z' = gz\bar{g}^{-1}$ for some $g \in G(\mathbb{C})$. Here the bar denotes the complex conjugation in $G(\mathbb{C})$; note that $G(\mathbb{R}) = \{g \in G(\mathbb{C}) \mid \bar{g} = g\}$. By definition, the neutral element $[1] \in H^1(\mathbb{R}, G)$ is the class of the neutral cocycle $1 \in Z^1(\mathbb{R}, G) \subset G(\mathbb{C})$.

We write $G(\mathbb{R})_2$ for the set of elements $g \in G(\mathbb{R})$ such that $g^2 = 1$. Then $g\bar{g} = g^2 = 1$, hence $G(\mathbb{R})_2 \subset Z^1(\mathbb{R}, G)$, and so we obtain a canonical map $G(\mathbb{R})_2 \rightarrow H^1(\mathbb{R}, G)$.

Let G be a simply connected absolutely simple \mathbb{R} -group. For simplicity we assume in the Introduction that G is an *inner form of a compact group*. Then G has a compact maximal torus T , see Subsection 3.1. Choose a basis Π of the root system $R = R(G_{\mathbb{C}}, T_{\mathbb{C}})$. We obtain an isomorphism

$$\gamma: L(D) \xrightarrow{\sim} T(\mathbb{R})_2 \subset Z^1(\mathbb{R}, G),$$

see formula (3) in Section 2. This isomorphism induces a map $L(D) \rightarrow H^1(\mathbb{R}, G)$, which is surjective by a result of Kottwitz [Ko86, Lemma 10.2]. By Theorem 3.2 the fibers of this map are equivalence classes of the Reeder puzzle for (D, \mathbf{t}) for a certain coloring \mathbf{t} of D . In other words, we obtain a bijection

$$\text{Cl}(D, \mathbf{t}) \xrightarrow{\sim} H^1(\mathbb{R}, G).$$

Moreover, for a suitable basis Π the coloring \mathbf{t} can be obtained from a Kac diagram by removing vertex 0, see Section 3.2. In Sections 6–14 we solve case by case the generalized Reeder puzzles for all such pairs (D, \mathbf{t}) . Namely, in each case we give a set Ξ of representatives for all equivalence classes in $\text{Cl}(D, \mathbf{t})$ and describe explicitly the equivalence class $[0] \subset L(D, \mathbf{t})$ of the zero labeling 0.

Now let H be a simply connected semisimple \mathbb{R} -subgroup of a simply connected absolutely simple \mathbb{R} -group G . For simplicity, we assume in the Introduction that H is absolutely simple and that G and H are inner forms of compact groups. Then G contains a compact

maximal torus T_G and H contains a compact maximal torus T_H . We may and shall assume that $T_H \subset T_G$. We denote by (D_H, \mathbf{t}_H) and (D_G, \mathbf{t}_G) the corresponding Reeder puzzles. For a good choice of bases Π_H and Π_G we obtain colorings \mathbf{t}_H and \mathbf{t}_G coming from Kac diagrams (in particular, not more than one vertex of each of D_H and D_G is black).

We describe our method of answering Question 0.2. The embedding $T_H \hookrightarrow T_G$ induces an embedding $T_H(\mathbb{R})_2 \hookrightarrow T_G(\mathbb{R})_2$. Thus we obtain an injective homomorphism

$$\iota: L(D_H) \rightarrow L(D_G),$$

which can be computed explicitly.

Our method works as follows. Using results of Sections 6–14 for the group H , we construct a finite subset $\Xi \subset L(D_H, \mathbf{t}_H)$ containing exactly one representative of each equivalence class for the corresponding Reeder puzzle. In other words, the composite map $\Xi \hookrightarrow L(D_H, \mathbf{t}_H) \rightarrow \text{Cl}(D_H, \mathbf{t}_H)$, see the commutative diagram below, is bijective. For any $\xi \in \Xi \subset L(D_H, \mathbf{t}_H)$ we compute $\iota(\xi) \in L(D_G, \mathbf{t}_G)$. Using results of Sections 6–14 for the group G , namely, the description of the equivalence class $[0]$ of 0 in $L(G, \mathbf{t}_G)$, we can check whether $\iota(\xi) \in L(D_G, \mathbf{t}_G)$ lies in $[0]$ or not. We obtain a subset Ξ_0 of Ξ consisting of all $\xi \in \Xi$ such that $\iota(\xi) \in [0]$. Then $\Xi_0 \subset \Xi$ is the preimage of $[0] \in \text{Cl}(D_G, \mathbf{t}_G)$ under the map $\Xi \hookrightarrow L(D_H, \mathbf{t}_H) \rightarrow \text{Cl}(D_G, \mathbf{t}_G)$, see the commutative diagram:

$$\begin{array}{ccccccc} \Xi & \longrightarrow & L(D_H, \mathbf{t}_H) & \longrightarrow & \text{Cl}(D_H, \mathbf{t}_H) & \xrightarrow{\sim} & H^1(\mathbb{R}, H) \\ & & \downarrow \iota & & \downarrow & & \downarrow \\ & & L(D_G, \mathbf{t}_G) & \longrightarrow & \text{Cl}(D_G, \mathbf{t}_G) & \xrightarrow{\sim} & H^1(\mathbb{R}, G) \end{array}$$

It follows that Ξ_0 is in a bijection with $\ker [H^1(\mathbb{R}, H) \rightarrow H^1(\mathbb{R}, G)]$, and therefore, the cardinality of Ξ_0 answers Questions 0.2 and 0.1.

Note that in order to answer Question 0.2, we compute in Sections 6–14 the sets $\text{Cl}(D, \mathbf{t})$ for inner forms of a compact group and certain sets $\text{Cl}(D, \tau, \mathbf{t})$ for outer forms. Since each of these sets is in a bijection with the corresponding Galois cohomology set, we, in particular, compute the cardinalities of the Galois cohomology sets $H^1(\mathbb{R}, H)$ for all absolutely simple simply connected \mathbb{R} -groups H . These cardinalities have been known. The Galois cohomology of classical groups and adjoint groups is well known. Garibaldi and Semenov [GS10, Example 5.1] computed $H^1(\mathbb{R}, H)$ for a certain nonsplit simply connected group H of type \mathbf{E}_7 . Conrad [Co14, Proof of Lemma 4.9] computed $H^1(\mathbb{R}, H)$ for the split simply connected groups H of types of \mathbf{E}_6 and \mathbf{E}_7 . The cardinalities of the Galois cohomology sets for “most” of simple \mathbb{R} -groups, in particular, for all absolutely simple simply connected \mathbb{R} -groups, were recently computed by Adams [A13] by a method different from ours. Our results agree with the previous results. Later, after the first version of the present paper appeared in arXiv, Borovoi and Timashev [BT15] proposed a combinatorial method based on the notion of a Kac diagram, permitting one to compute easily the cardinality of $H^1(\mathbb{R}, H)$ when H is an inner form of any compact semisimple \mathbb{R} -group, not necessarily simply connected. However, it seems that neither of these alternative approaches permits one to answer Question 0.1 about $(G/H)(\mathbb{R})$, except for the case when $H^1(\mathbb{R}, G) = 1$ (which happens only when $G = \mathbf{SL}(n)$ or $G = \mathbf{Sp}(2n)$).

The rest of the paper is structured as follows. In Section 1 we recall results of [Bo88]. In Sections 2 and 3 we compute the moves \mathcal{M}_i in the case when G is compact and when it is a noncompact inner form of a compact group, respectively. In particular, in Section 3 we prove Theorem 3.2 describing the pointed set $H^1(\mathbb{R}, G)$ for an *inner* form G of a compact simply connected simple group in terms of the corresponding generalized Reeder puzzle. In Section 4 we prove Theorem 4.4, which reduces computing the Galois cohomology of an *outer* form of a compact, simply connected, simple \mathbb{R} -group to computing Galois cohomology of an *inner* form of another compact group. In Sections 6–14, using Theorems 3.2 and 4.4, we solve the generalized Reeder puzzles for all isomorphism classes

of simply connected absolutely simple \mathbb{R} -groups G . We state the assertions necessary for our calculations, but omit straightforward proofs for brevity. In the last Section 15 we describe our method of answering Questions 0.2 and 0.1 for all simply connected H (not necessarily simple), and we give examples of calculations using results of Sections 6–14.

1. GALOIS COHOMOLOGY OF REDUCTIVE REAL GROUPS

In this section we state briefly the necessary results of [Bo88]. For details see [Bo88] or [Bo14].

Let G be a reductive group over \mathbb{R} . Let T be a *fundamental torus* of G , i.e., a maximal torus of G containing a maximal compact torus T_0 of G . Then T is the centralizer of T_0 in G ; see [Bo14, Section 7]. Let T_1 be the largest *split* subtorus of T . We write $T(\mathbb{R})_2$ for the group of elements of $T(\mathbb{R})$ of order dividing 2.

Lemma 1.1 ([Bo88, Lemma 1.1], see also [Bo14, Lemma 3(a)]). *The map $T(\mathbb{R})_2 \rightarrow H^1(\mathbb{R}, T)$ induces a canonical isomorphism $T(\mathbb{R})_2/T_1(\mathbb{R})_2 \xrightarrow{\sim} H^1(\mathbb{R}, T)$.*

Set $N_0 = \mathcal{N}_G(T_0)$, $W_0 = N_0/T$. We have $W_0(\mathbb{C}) = W_0(\mathbb{R})$; see [Bo14, Section 7]. We define a left action of the group $W_0(\mathbb{R})$ on the set $H^1(\mathbb{R}, T)$. Let $w \in W_0(\mathbb{R})$ be represented by $n \in N_0(\mathbb{C})$ and let $\xi \in H^1(\mathbb{R}, T)$, $\xi = [z]$, where $z \in Z^1(\mathbb{R}, T)$ is a cocycle and $[z]$ denotes the cohomology class of z . We set

$$(2) \quad w * \xi := [nz\bar{n}^{-1}] = [nzn^{-1} \cdot n\bar{n}^{-1}],$$

where the bar denotes the complex conjugation in $G(\mathbb{C})$. This is a well-defined action; see [Bo14, Construction 8]. (Note that in general the action $*$ does not respect the group structure on $H^1(\mathbb{R}, T)$.) It is easy to see that the images of ξ and $w * \xi$ in $H^1(\mathbb{R}, G)$ coincide. Therefore, we obtain a canonical map

$$W_0(\mathbb{R}) \backslash H^1(\mathbb{R}, T) \rightarrow H^1(\mathbb{R}, G).$$

Proposition 1.2 ([Bo88, Theorem 1], see also [Bo14, Theorem 9]). *The map*

$$W_0(\mathbb{R}) \backslash H^1(\mathbb{R}, T) \rightarrow H^1(\mathbb{R}, G)$$

induced by the map $H^1(\mathbb{R}, T) \rightarrow H^1(\mathbb{R}, G)$ is a bijection.

2. WEYL ACTION FOR COMPACT GROUPS

We change our notation. In Sections 2, 3, and 4, G is a *simply connected, simple, compact* (i.e., anisotropic) linear algebraic group over \mathbb{R} .

Let T be a maximal torus of G . Let $X^* = X^*(T_{\mathbb{C}}) := \text{Hom}(T_{\mathbb{C}}, \mathbb{G}_{m, \mathbb{C}})$ denote the character group of $T_{\mathbb{C}}$, where $\mathbb{G}_{m, \mathbb{C}}$ is the multiplicative group over \mathbb{C} . Let $R = R(G_{\mathbb{C}}, T_{\mathbb{C}}) \subset X^*$ denote the root system of $G_{\mathbb{C}}$ with respect to $T_{\mathbb{C}}$, and let $\Pi \subset R$ be a basis of R (a system of simple roots). Note that Π does not have to be a basis of X^* . Write $\Pi = \{\alpha_1, \dots, \alpha_n\}$, then a simple root α_i is a homomorphism $\alpha_i: T_{\mathbb{C}} \rightarrow \mathbb{G}_{m, \mathbb{C}}$. Let $W = W(G, T) = N/T$ denote the Weyl group, where N is the normalizer of T in G . By abuse of notation we write W also for the group of points $W(\mathbb{R}) = W(\mathbb{C})$. Let $D = D(G_{\mathbb{C}}, T_{\mathbb{C}}, \Pi)$ denote the Dynkin diagram of $G_{\mathbb{C}}$ with respect to $T_{\mathbb{C}}$ and Π , then the set of vertices of D is Π .

Let $X_* = X_*(T_{\mathbb{C}}) := \text{Hom}(\mathbb{G}_{m, \mathbb{C}}, T_{\mathbb{C}})$ denote the cocharacter group of T . There is a canonical pairing

$$\langle \cdot, \cdot \rangle: X^* \times X_* \rightarrow \mathbb{Z}, \quad (\chi, x) \mapsto \langle \chi, x \rangle \in \mathbb{Z}, \quad \chi \in X^*, \quad x \in X_*$$

defined by

$$\chi \circ x = (z \mapsto z^{\langle \chi, x \rangle}): \mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbb{G}_{m, \mathbb{C}}.$$

We have a canonical basis $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ of the dual root system R^\vee , where the simple coroot $\alpha_i^\vee: \mathbb{G}_{m, \mathbb{C}} \rightarrow T_{\mathbb{C}}$ is the coroot corresponding to the simple root α_i ; see

[Sp98, Sections 7.4 and 7.5]. Note that $\langle \alpha_i, \alpha_i^\vee \rangle = 2$. Since G is *simply connected*, Π^\vee is a basis of X_* ; cf. [Sp98, Section 8.1.11].

We are interested in the action of W in $T(\mathbb{R})_2$. We identify $X_*/2X_*$ with $T(\mathbb{R})_2$ by $x + 2X_* \mapsto x(-1) \in T(\mathbb{R})_2$ for $x \in X_*$. The canonical \mathbb{Z} -basis $\alpha_1^\vee, \dots, \alpha_n^\vee$ of X_* gives a $\mathbb{Z}/2\mathbb{Z}$ -basis of $X_*/2X_*$, which we shall again write as $\alpha_1^\vee, \dots, \alpha_n^\vee$.

By a *labeling* of the Dynkin diagram D we mean a vector $\mathbf{a} = (a_i)_{i=1, \dots, n}$, where $a_i \in \mathbb{Z}/2\mathbb{Z}$, i.e., $a_i = 0, 1$. In other words, at each vertex i of D we write a label $a_i \in \mathbb{Z}/2\mathbb{Z}$. We denote the abelian group of labelings of D by $L(D)$. We have a canonical isomorphism

$$(3) \quad \gamma: L(D) \xrightarrow{\sim} T(\mathbb{R})_2 \subset Z^1(\mathbb{R}, G), \quad \mathbf{a} \mapsto a = \prod_{i=1}^n (\alpha_i^\vee(-1))^{a_i}.$$

By abuse of notation we denote by γ both the isomorphism $\gamma: L(D) \xrightarrow{\sim} T(\mathbb{R})_2$ and the embedding $\gamma: L(D) \xrightarrow{\sim} T(\mathbb{R})_2 \hookrightarrow Z^1(\mathbb{R}, G)$. Thus with $\mathbf{a} \in L(D)$ we associate $a = \gamma(\mathbf{a}) \in T(\mathbb{R})_2 \subset Z^1(\mathbb{R}, G)$. We also associate with \mathbf{a} the element $\sum_k a_k \alpha_k^\vee \in X_*/2X_*$.

We wish to compute the orbits of W in $T(\mathbb{R})_2$ with respect to the canonical left action. The Weyl group W is generated by the reflections $r_i = r_{\alpha_i}$. We define the *moves* $\mathcal{M}_i: L(D) \rightarrow L(D)$ on the set of labelings $L(D)$ by $\mathcal{M}_i \mathbf{a} = \mathbf{a}'$, where

$$(4) \quad r_i \left(\prod_{j=1}^n (\alpha_j^\vee(-1))^{a_j} \right) = \prod_{j=1}^n (\alpha_j^\vee(-1))^{a'_j} \quad \text{i.e.,} \quad r_i \left(\sum_{j=1}^n a_j \alpha_j^\vee \right) = \sum_{j=1}^n a'_j \alpha_j^\vee.$$

Note that if $\mathbf{a}' = \mathcal{M}_i \mathbf{a}$, then $\mathbf{a} = \mathcal{M}_i \mathbf{a}'$, because $r_i^2 = 1$. We say that two labelings $\mathbf{a}, \mathbf{a}' \in L(D)$ are *equivalent* if we can relate them by a series of moves. The set of orbits of W in $T(\mathbb{R})_2$ is in a canonical bijection with the set of equivalence classes of labelings $\mathbf{a} \in L(D)$ of the Dynkin diagram D of $(G_{\mathbb{C}}, T_{\mathbb{C}}, \Pi)$ with respect to the moves.

The following Lemma 2.1 says that the moves defined in this sections are indeed the moves of the Reeder puzzle on D .

Lemma 2.1. *Let G be a simply connected, simple, compact \mathbb{R} -group of absolute rank n , and D its Dynkin diagram, as above. Define the moves $\mathcal{M}_i: L(D) \rightarrow L(D)$ by (4). Then we have $a'_j = a_j$ for $j \neq i$, and a'_i is given by*

$$(5) \quad a'_i = a_i + \sum'_{k \succeq i} a_k$$

(addition in $\mathbb{Z}/2\mathbb{Z}$), where $\sum'_{k \succeq i}$ means that the sum is taken over all the neighbors $k \neq i$ of i except for the vertices k connected to i by a double edge such that the root α_k is shorter than α_i .

Proof. A reflection r_i acts on X_* by

$$(6) \quad r_i(y) = y - \langle \alpha_i, y \rangle \alpha_i^\vee,$$

cf. [Sp98, Section 7.4.1]. If $y = \sum_k a_k \alpha_k^\vee \in X_*$, then

$$r_i(y) = y - \sum_k a_k \langle \alpha_i, \alpha_k^\vee \rangle \alpha_i^\vee,$$

and the same formula holds if $y = \sum_k a_k \alpha_k^\vee \in X_*/2X_*$. If we write $r_i(y) = \sum_k a'_k \alpha_k^\vee$, then clearly $a'_j = a_j$ for $j \neq i$, and

$$(7) \quad a'_i = a_i + \sum_k (-a_k) \langle \alpha_i, \alpha_k^\vee \rangle,$$

so we need only to compute (in $\mathbb{Z}/2\mathbb{Z}$) the sum in (7).

We assume that our root system R is a root system in a Euclidean space V . Then

$$\langle \alpha_i, \alpha_k^\vee \rangle = \frac{2(\alpha_i, \alpha_k)}{(\alpha_k, \alpha_k)},$$

where (α_i, α_k) is the scalar product in V . If $k = i$, then $\langle \alpha_i, \alpha_k^\vee \rangle = \langle \alpha_i, \alpha_i^\vee \rangle = 2 \equiv 0 \pmod{2}$. If two different vertices i and k are not connected by an edge, then $\langle \alpha_i, \alpha_k^\vee \rangle = 0$. Thus the sum in (7) is taken over vertices k different from i that are connected to i by an edge. Now we consider cases. If vertices i and k are connected by a single edge, then $\langle \alpha_i, \alpha_k^\vee \rangle = -1$ [Bou68, VI.1.3, possibility (3)], hence vertex k gives a_k to the sum in (7). If they are connected by a triple edge, then either $\langle \alpha_i, \alpha_k^\vee \rangle = -1$ or $\langle \alpha_i, \alpha_k^\vee \rangle = -3 \equiv -1 \pmod{2}$ [Bou68, VI.1.3, possibility (7)], and again vertex k gives a_k to the sum. If they are connected by a double edge and the root α_k is *longer* than α_i , then $\langle \alpha_i, \alpha_k^\vee \rangle = -1$ [Bou68, VI.1.3, possibility (5)], and again vertex k gives a_k to the sum. However, if the vertices i and k are connected by a double edge and the root α_k is *shorter* than α_i , then $\langle \alpha_i, \alpha_k^\vee \rangle = -2 \equiv 0 \pmod{2}$ [Bou68, VI.1.3, possibility (5)], hence vertex k gives nothing to the sum in (7). We conclude that formula (7) can be written as (5). \square

3. WEYL ACTION FOR INNER FORMS

In this section G , T , R , Π , D , and W are as in Section 2, in particular G is a simply connected, simple, *compact* linear algebraic group over \mathbb{R} .

3.1. The t -twisted action. Write $G^{\text{ad}} = G/Z_G$, $T^{\text{ad}} = T/Z_G$, where Z_G denotes the center of G . Then T^{ad} is a maximal torus in the adjoint group G^{ad} . Consider an inner twisted form (inner twist) ${}_zG$ of G , where $z \in Z^1(\mathbb{R}, G^{\text{ad}})$. It is well known that z is cohomologous to some $t \in T^{\text{ad}}(\mathbb{R})_2$ (see e.g., [Se94, III.4.5, Example (a)]). We fix such an element t . Then ${}_zG \simeq {}_tG$. We have ${}_tG(\mathbb{C}) = G(\mathbb{C})$, but the complex conjugation in ${}_tG(\mathbb{C})$ is given by

$$g \mapsto {}^*\bar{g} = \text{Inn}(t)(\bar{g}).$$

This means that if we lift $t \in T^{\text{ad}}(\mathbb{R})_2$ to some $\tilde{t} \in T(\mathbb{C})$, then the complex conjugation in ${}_tG(\mathbb{C})$ is given by

$${}^*\bar{g} = \tilde{t} \bar{g} \tilde{t}^{-1}.$$

Since $\tilde{t} \in T(\mathbb{C})$, we have ${}_tT = T$, hence ${}_tT$ is a compact maximal torus in ${}_tG$, hence it is a fundamental torus of ${}_tG$. Thus any inner form of a compact semisimple \mathbb{R} -group has a compact maximal torus. Let T_0 of Section 1 be the maximal compact subtorus of ${}_tT$, then clearly $T_0 = {}_tT = T$. Let $W_0 := W_0({}_tG, {}_tT)$ be the group W_0 of Section 1, then $W_0 = W(G, T) = W$, because W_0 was defined in terms of T_0 .

We consider the t -twisted action of W given by formula (2) on $H^1(\mathbb{R}, {}_tT) = H^1(\mathbb{R}, T) = T(\mathbb{R})_2$. Let $w \in W(\mathbb{R}) = W(\mathbb{C})$, $w = nT$, where $n \in N(\mathbb{R})$. For $a \in T(\mathbb{R})_2 = T(\mathbb{C})_2$ the t -twisted action of w is given by

$$(8) \quad w * a = n a {}^*\bar{n}^{-1} = n a \tilde{t} \bar{n}^{-1} \tilde{t}^{-1} = n a \tilde{t} n^{-1} \tilde{t}^{-1} = n a n^{-1} \cdot n \tilde{t} n^{-1} \tilde{t}^{-1}.$$

In particular, let $r_j \in W(\mathbb{R}) = W(\mathbb{C})$ be the reflection corresponding to a simple root α_j . Write $r_j = n_j T$ for some $n_j \in N(\mathbb{R})$. For $a \in T(\mathbb{R})_2$ the t -twisted action of r_j is given by

$$(9) \quad r_j * a = n_j a {}^*\bar{n}_j^{-1} = n_j a \tilde{t} n_j^{-1} \tilde{t}^{-1} = n_j a n_j^{-1} \cdot n_j \tilde{t} n_j^{-1} \tilde{t}^{-1}.$$

Note that

$$(10) \quad r_j * a = r_j(a) \cdot n_j \tilde{t} n_j^{-1} \tilde{t}^{-1},$$

where $r_j(a) = n_j a n_j^{-1}$. In particular, we have $r_j * 1 = n_j \tilde{t} n_j^{-1} \tilde{t}^{-1}$, so in general $r_j * 1 \neq 1$ and therefore, the t -twisted action does not preserve the group structure in $T(\mathbb{R})_2$.

Define

$$(11) \quad \mathbf{t} = (t_i) \in (\mathbb{Z}/2\mathbb{Z})^n, \quad \text{where} \quad (-1)^{t_i} = \alpha_i(t).$$

We regard \mathbf{t} as a *coloring* of the diagram D . We color a vertex i in black if $t_i = 1$, and leave i uncolored (i.e., white) if $t_i = 0$. Denote by ${}_{\mathbf{t}}D := (D, \mathbf{t})$ the Dynkin diagram $D = D(G_{\mathbb{C}}, T_{\mathbb{C}}, \Pi)$ together with the coloring \mathbf{t} . The notation ${}_{\mathbf{t}}D$ suggests that we regard ${}_{\mathbf{t}}D = (D, \mathbf{t})$ as an (inner) twist of D by \mathbf{t} .

We compute the moves corresponding to the t -twisted action. For each vertex i of D , we define the move \mathcal{M}_i by $\mathcal{M}_i \mathbf{a} = \mathbf{a}'$, where

$$r_i * \left(\prod_{j=1}^n (\alpha_j^\vee(-1))^{a_j} \right) = \prod_{j=1}^n (\alpha_j^\vee(-1))^{a'_j} \quad \text{i.e.,} \quad r_i * \left(\sum_{j=1}^n a_j \alpha_j^\vee \right) = \sum_{j=1}^n a'_j \alpha_j^\vee.$$

Lemma 3.1. *For the t -twisted action of W and the move \mathcal{M}_i just defined, we have, as in Lemma 2.1, $a'_j = a_j$ for $j \neq i$, while in formula (5) the term $t_i \in \mathbb{Z}/2\mathbb{Z}$ defined by $(-1)^{t_i} = \alpha_i(t)$ must be added. Thus we have*

$$(12) \quad a'_i = a_i + t_i + \sum'_{k \succeq i} a_k,$$

where the meaning of $\sum'_{k \succeq i}$ is the same as in formula (5).

Proof. By (10) and Lemma 2.1 it suffices to show that $n_j \tilde{t} n_j^{-1} \tilde{t}^{-1} = (\alpha_j^\vee(-1))^{t_j}$. We are indebted to Dmitry A. Timashev for the idea of the following proof.

Consider the \mathbb{C} -torus $T_{\mathbb{C}}$. As above, we write X_* for $\mathbf{X}_*(T_{\mathbb{C}}) = \text{Hom}(\mathbb{G}_{m, \mathbb{C}}, T_{\mathbb{C}})$. We have a canonical isomorphism of abelian complex Lie groups

$$X_* \otimes_{\mathbb{Z}} \mathbb{C}^\times \xrightarrow{\sim} T(\mathbb{C}), \quad x \otimes u \mapsto x(u), \quad x \in X_*, \quad u \in \mathbb{C}^\times = \mathbb{G}_{m, \mathbb{C}}(\mathbb{C}).$$

Thus we obtain an isomorphism of abelian complex Lie algebras (vector spaces over \mathbb{C})

$$X_* \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \text{Lie } T_{\mathbb{C}}, \quad x \otimes v \mapsto dx(v), \quad x \in X_*, \quad v \in \mathbb{C}, \quad dx := d_1 x : \mathbb{C} = \text{Lie } \mathbb{G}_{m, \mathbb{C}} \rightarrow \text{Lie } T_{\mathbb{C}}.$$

In particular, we obtain a canonical embedding

$$(13) \quad X_* \hookrightarrow X_* \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \text{Lie } T_{\mathbb{C}} \quad x \mapsto x \otimes 1 \mapsto dx(1).$$

Now it is an easy exercise to deduce from (6) that for $1 \leq j \leq n$ and for any $y \in \text{Lie } T_{\mathbb{C}}$ we have

$$(14) \quad r_j(y) = y - \langle d\alpha_j, y \rangle d\alpha_j^\vee(1),$$

where we write $d\alpha_j$ for $d_1 \alpha_j : \text{Lie } T_{\mathbb{C}} \rightarrow \text{Lie } \mathbb{G}_{m, \mathbb{C}} = \mathbb{C}$, and we write $\langle d\alpha_j, y \rangle$ for $d\alpha_j(y) \in \mathbb{C}$.

Let $\omega_k^\vee \in \text{Lie } T_{\mathbb{C}}$ be the element such that $\langle d\alpha_j, \omega_k^\vee \rangle = \delta_{jk}$, where δ_{jk} is Kronecker's delta symbol. We set $\tilde{t} = \exp(\pi \mathbf{i} \sum_k t_k \omega_k^\vee) \in T(\mathbb{C})$, where $\mathbf{i}^2 = -1$. Then

$$\alpha_j(\tilde{t}) = \exp \left\langle d\alpha_j, \pi \mathbf{i} \sum_k t_k \omega_k^\vee \right\rangle = \exp \left(\pi \mathbf{i} \sum_k t_k \langle d\alpha_j, \omega_k^\vee \rangle \right) = \exp(\pi \mathbf{i} t_j) = (-1)^{t_j},$$

because the exponential map commutes with homomorphisms of Lie groups; see [OV90, Section 1.2.7, p. 29, Problem 26]. It follows that the image of \tilde{t} in $T^{\text{ad}}(\mathbb{C})$ is indeed t . By (14) we have

$$\begin{aligned} n_j \tilde{t} n_j^{-1} \tilde{t}^{-1} &= r_j(\tilde{t}) \tilde{t}^{-1} = \exp \left(\pi \mathbf{i} \sum_k t_k (r_j(\omega_k^\vee) - \omega_k^\vee) \right) \\ &= \exp \left(-\pi \mathbf{i} \sum_k t_k \langle d\alpha_j, \omega_k^\vee \rangle d\alpha_j^\vee(1) \right) = \exp(t_j d\alpha_j^\vee(-\pi \mathbf{i})) = (\alpha_j^\vee(-1))^{t_j}. \end{aligned}$$

Thus $n_j \tilde{t} n_j^{-1} \tilde{t}^{-1} = (\alpha_j^\vee(-1))^{t_j}$, as required. \square

According to Lemma 3.1, the twisted action of \mathcal{M}_i on a labeling $\mathbf{a} = (a_i) \in (\mathbb{Z}/2\mathbb{Z})^n$ is given by formula (12). This means that for any vertex i of D , the action of \mathcal{M}_i is given by formula (5) if vertex i is white (i.e., $t_i = 0$), and by formula

$$(15) \quad a'_i = a_i + 1 + \sum'_{k \succeq i} a_k,$$

if vertex i is black (i.e., $t_i = 1$). In other words, this is exactly the generalized Reader puzzle as described in the Introduction. We denote by $L(D, \mathbf{t})$ (or $L(\mathbf{t}D)$) the set of labelings $(\mathbb{Z}/2\mathbb{Z})^n$ with this twisted action of the moves \mathcal{M}_i . By Lemma 3.1 the action of \mathcal{M}_i on $L(D, \mathbf{t})$ is compatible with the t -twisted action of the reflection $r_i \in W$ on $T(\mathbb{R})_2 = {}_tT(\mathbb{R})_2$ with respect to the canonical bijection

$$\gamma_{\mathbf{t}}: L(D, \mathbf{t}) \xrightarrow{\sim} T(\mathbb{R})_2 \subset Z^1(\mathbb{R}, {}_tG) \quad \mathbf{a} \mapsto a = \prod_i (\alpha_i^\vee(-1))^{a_i}.$$

By abuse of notation we denote by $\gamma_{\mathbf{t}}$ both the isomorphism $\gamma_{\mathbf{t}}: L(D) \xrightarrow{\sim} T(\mathbb{R})_2$ and the embedding $\gamma_{\mathbf{t}}: L(D) \xrightarrow{\sim} T(\mathbb{R})_2 \hookrightarrow Z^1(\mathbb{R}, G)$. We regard the twisted diagram $\mathbf{t}D = (D, \mathbf{t})$ as the *colored Dynkin diagram of the twisted group* ${}_tG$ (with respect to T and Π). We denote by $\text{Orb}(\mathbf{t}D)$ the set of equivalence classes (orbits) in $L(\mathbf{t}D)$ with respect to the equivalence relation given by the moves of Lemma 3.1 (in the Introduction we denoted this set of equivalence classes by $\text{Cl}(D, \mathbf{t})$).

The following theorem describes the Galois cohomology of an *inner* form ${}_tG$ of a compact, simply connected, simple \mathbb{R} -group G in terms of labelings of the corresponding colored Dynkin diagram $\mathbf{t}D$.

Theorem 3.2. *Let G , T , R , Π , D , and W be as in Section 2. Let $t \in T^{\text{ad}}(\mathbb{R})_2$ and let $\mathbf{t} \in (\mathbb{Z}/2\mathbb{Z})^n$ be defined by (11). Let $L(\mathbf{t}D)$ be the set of labelings of the colored Dynkin diagram $\mathbf{t}D$ with the moves given by formula (12). Then the canonical map*

$$\gamma_{\mathbf{t}}: L(\mathbf{t}D) \xrightarrow{\sim} T(\mathbb{R})_2 \hookrightarrow Z^1(\mathbb{R}, {}_tG)$$

induces a canonical bijection

$$\lambda_{\mathbf{t}}: \text{Orb}(\mathbf{t}D) \xrightarrow{\sim} H^1(\mathbb{R}, {}_tG).$$

The theorem follows immediately from Proposition 1.2 and Lemma 3.1.

3.2. The Kac diagram, twisting diagram, and augmented diagram. An inner form ${}_tG$ of G is given by an element $t \in G^{\text{ad}}(\mathbb{R})_2$. By Kac [Kac69], see also Helgason [H78, Ch. X, § 5] and Onishchik and Vinberg [OV90, Ch. 4, § 4 and Ch. 5, § 1], the pairs (G, t) , where G is a simply connected, simple, compact \mathbb{R} -group, and $t \in G^{\text{ad}}(\mathbb{R})_2$ is an involutive inner automorphism of G , can be described (up to an isomorphism of pairs) using the Kac diagrams of types I and II in [OV90, Table 7]. (The Kac diagrams of type III in [OV90, Table 7] correspond to involutive *outer* automorphisms.) The relation between Kac diagrams and pairs (G, t) is as follows. Let Δ be a Kac diagram of type I or II in [OV90, Table 7]. It is the extended Dynkin diagram of $(G_{\mathbb{C}}, T_{\mathbb{C}}, \Pi)$ for some G, T, Π as in Section 2, and some of the vertices of this diagram are colored in black, while the others are white. The underlying extended Dynkin diagram has $n + 1$ vertices corresponding to the roots $\alpha_0, \alpha_1, \dots, \alpha_n$, where $\alpha_1, \dots, \alpha_n$ are the simple roots and α_0 is the lowest root. If the Kac diagram Δ is of type I, then it has exactly one black vertex, say vertex i , where $i \neq 0$. If Δ is of type II, then it has exactly two black vertices, vertex 0 and another black vertex, say vertex i , where again $i \neq 0$. We set $\nu_j = 0$ if vertex j is white, and set $\nu_j = 1$ if vertex j is black. Let m_j ($j = 0, 1, \dots, n$) denote the coefficients of linear dependence

$$m_0\alpha_0 + m_1\alpha_1 + \dots + m_n\alpha_n = 0$$

normalized so that m_j are relatively prime positive integers (then $m_0 = 1$), see [OV90, Table 6]. Then the numbers ν_j satisfy

$$(16) \quad m_0\nu_0 + m_1\nu_1 + \cdots + m_n\nu_n = 2,$$

see [OV90, Section 5.1.5, formula (12)], or [OV94, Section 3.3.7, formula (24)]. The Kac diagram Δ determines an element $t = t_\Delta \in T^{\text{ad}}(\mathbb{R})_2$ such that

$$(17) \quad \alpha_j(t) = (-1)^{\nu_j}, \quad j = 1, \dots, n.$$

This element t defines an inner twisted form $G_\Delta := {}_tG$ of G . Define $\mathbf{t} = (t_j)_{j=1, \dots, n}$ by formula (11). Comparing formulas (11) and (17), we obtain that

$$(18) \quad (-1)^{t_j} := \alpha_j(t) = (-1)^{\nu_j} \quad \text{for } j = 1, \dots, n.$$

Thus \mathbf{t} is the coloring of the Dynkin diagram D of $G_{\mathbb{C}}$ such that vertex i is black and the other vertices are white.

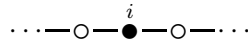
Let ${}_zG$ be an inner twist of G , where $z \in Z^1(\mathbb{R}, G^{\text{ad}})$. We have ${}_zG \simeq {}_tG$ for some $t \in T^{\text{ad}}(\mathbb{R})_2$. By [OV94, Section 4.1.4], if $t \neq 1$, then up to an automorphism of G , the element t comes from a unique Kac diagram Δ of type I or II from [OV90, Table 7], where t is determined by Δ via formula (17). We call the *twisting diagram* of ${}_zG$ the colored Dynkin diagram (D, \mathbf{t}) defined in the previous paragraph. By (18) we obtain the twisting diagram of ${}_zG$ by removing vertex 0 from the Kac diagram Δ . The twisting diagram has exactly one black vertex, say vertex i . We say that i is the *twisting vertex* for the twisted by t action of W . We also say that the action of W is *twisted at i* . The next lemma follows immediately from Lemma 3.1.

Lemma 3.3. *For the twisted at i action of W , we have the same formula (5) for \mathcal{M}_j for $j \neq i$ as in Lemma 2.1. For \mathcal{M}_i we have, as in Lemma 2.1, $a'_j = a_j$ for $j \neq i$, while in formula (5) for a'_i we must add 1. Namely, we have*

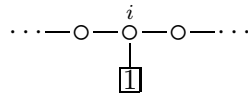
$$(19) \quad a'_i = a_i + 1 + \sum'_{k \succeq i} a_k,$$

where the meaning of $\sum'_{k \succeq i}$ is the same as in formula (5).

Construction 3.4. Assume we have a twisting diagram with a black vertex i :

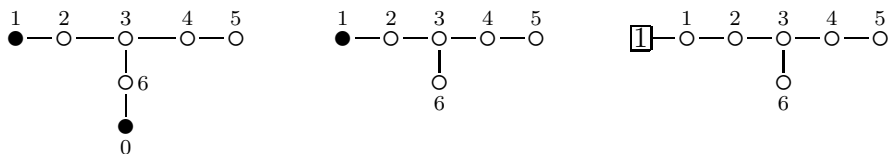


We have formula (19) for \mathcal{M}_i . In order to get formula (5) instead, we color the vertex i in white and *augment* our diagram by formally adding a new vertex which we call the *boxed 1*, connected by a simple edge to vertex i :



Here 1 in the box means that we put 1 as the label at this new vertex. Now the formula for \mathcal{M}_i becomes (5) where the boxed 1 is included in the sum. Thus this boxed 1 accounts for twisting at i . Note that we do not add a move corresponding to the boxed 1, so the label 1 at the boxed 1 cannot be changed by moves. We call the obtained diagram the *augmented diagram* corresponding to the twisting vertex i .

Example 3.5. These are the Kac diagram, the twisting diagram and the augmented diagram for the group E_{III} of type \mathbf{E}_6 (see Subsection 10.3 below):



4. WEYL ACTION FOR OUTER FORMS

In this section again G , T , R , Π , D , and W are as in Section 2, in particular G is a simply connected, simple, *compact* linear algebraic group over \mathbb{R} .

Let ${}_zG$ be an outer twisted form (outer twist) of G , where $z \in Z^1(\mathbb{R}, \text{Aut } G)$ and $z \notin Z^1(\mathbb{R}, \text{Inn } G)$. We have a canonical split epimorphism

$$\psi: \text{Aut } G \rightarrow \text{Aut } D,$$

see [Sp77, Corollary 2.14], and so we obtain an element $\tau = \psi_*(z) \in Z^1(\mathbb{R}, \text{Aut } D)$. It is easy to see that the complex conjugation acts trivially on $\text{Aut } D$, and therefore, $Z^1(\mathbb{R}, \text{Aut } D) = (\text{Aut } D)_2$. The involutive automorphism τ of D acts on Π , Π^\vee , T , G , and W . We write W^τ for the algebraic subgroup of fixed points of τ in W . We write ${}_\tau T$, ${}_\tau T^{\text{ad}}$, ${}_\tau G$, ${}_\tau G^{\text{ad}}$, and ${}_\tau W$ for the corresponding twisted algebraic groups. We write Π^τ for the set of fixed points of τ in Π , and D^τ for the corresponding Dynkin subdiagram.

We consider the action of τ on Π and on Π^\vee . The decomposition

$$\Pi = \Pi^\tau \cup (\Pi \setminus \Pi^\tau)$$

of the basis Π of the character group $X^*(T^{\text{ad}})$ of the adjoint torus T^{ad} induces a τ -invariant decomposition into a direct product

$$T^{\text{ad}} = T^{\text{ad}}(D^\tau) \times T^{\text{ad}}(D \setminus D^\tau)$$

with $X^*(T^{\text{ad}}(D^\tau)) = \langle \Pi^\tau \rangle$ and $X^*(T^{\text{ad}}(D \setminus D^\tau)) = \langle \Pi \setminus \Pi^\tau \rangle$. Here for a subset $S \subset X^*(T^{\text{ad}})$, we denote by $\langle S \rangle$ the subgroup generated by S . Concerning the corresponding τ -twisted tori, we see that ${}_\tau T^{\text{ad}}(D^\tau) = T^{\text{ad}}(D^\tau)$ is a compact torus, while the \mathbb{R} -torus ${}_\tau T^{\text{ad}}(D \setminus D^\tau)$ is isomorphic to the Weil restriction of scalars $R_{\mathbb{C}/\mathbb{R}} T'$ of some \mathbb{C} -torus T' . It follows that

$$H^1(\mathbb{R}, {}_\tau T^{\text{ad}}(D^\tau)) = T^{\text{ad}}(D^\tau)(\mathbb{R})_2, \quad \text{while} \quad H^1(\mathbb{R}, {}_\tau T^{\text{ad}}(D \setminus D^\tau)) = 1,$$

and therefore, the embedding ${}_\tau T^{\text{ad}}(D^\tau) \hookrightarrow {}_\tau T^{\text{ad}}$ induces a canonical isomorphism

$$(20) \quad T^{\text{ad}}(D^\tau)(\mathbb{R})_2 = H^1(\mathbb{R}, {}_\tau T^{\text{ad}}(D^\tau)) \xrightarrow{\sim} H^1(\mathbb{R}, {}_\tau T^{\text{ad}}).$$

Similarly, we have a decomposition

$$\Pi^\vee = \Pi^{\vee\tau} \cup (\Pi^\vee \setminus \Pi^{\vee\tau}),$$

where we write $\Pi^{\vee\tau}$ for $(\Pi^\vee)^\tau$. This decomposition of the basis Π^\vee of the cocharacter group $X_*(T)$ induces a τ -invariant decomposition into a direct product

$$T = T(D^\tau) \times T(D \setminus D^\tau)$$

with $X_*(T(D^\tau)) = \langle \Pi^{\vee\tau} \rangle$ and $X_*(T(D \setminus D^\tau)) = \langle \Pi^\vee \setminus \Pi^{\vee\tau} \rangle$. As above, we have ${}_\tau T(D^\tau) = T(D^\tau)$, hence

$$H^1(\mathbb{R}, {}_\tau T(D^\tau)) = T(D^\tau)(\mathbb{R})_2, \quad \text{while} \quad H^1(\mathbb{R}, {}_\tau T(D \setminus D^\tau)) = 1.$$

The map $\gamma: L(D) \rightarrow T(\mathbb{C})_2$ given by formula (3) induces an isomorphism $L(D)^\tau \rightarrow {}_\tau T(\mathbb{R})_2$. We obtain a commutative diagram

$$(21) \quad \begin{array}{ccccc} L(D^\tau) & \xrightarrow{\sim} & T(D^\tau)(\mathbb{R})_2 & \xrightarrow{\sim} & H^1(\mathbb{R}, T(D^\tau)) \\ \left(\uparrow \right) & & \left(\uparrow \right) & & \downarrow \sim \\ L(D)^\tau & \xrightarrow{\sim} & {}_\tau T(\mathbb{R})_2 & \longrightarrow & H^1(\mathbb{R}, {}_\tau T) \end{array}$$

with obvious maps.

Since $\tau = \psi_*(z)$, our outer form ${}_zG$ of G is an *inner* twist of ${}_\tau G$, i.e. ${}_zG \simeq {}_{z'}({}_\tau G)$ for some $z' \in Z^1(\mathbb{R}, {}_\tau G^{\text{ad}})$. By Proposition 1.2 the cocycle z' is cohomologous to some $t \in Z^1(\mathbb{R}, {}_\tau T^{\text{ad}})$, and by (20) we may assume that $t \in T^{\text{ad}}(D^\tau)(\mathbb{R})_2 \subset {}_\tau T^{\text{ad}}(\mathbb{R})_2 \subset Z^1(\mathbb{R}, {}_\tau G^{\text{ad}})$. We denote by $\text{Inn}(t)$ the corresponding inner automorphism of ${}_\tau G$ of order

dividing 2. We set $\sigma = \text{Inn}(t) \circ \tau$. Note that $\text{Inn}(t)$ and τ commute, hence σ is an outer automorphism of order 2 of G . We write ${}_{\sigma}G = {}_{\text{Inn}(t)}({}_{\tau}G)$ for the corresponding twisted form of G . For simplicity we also write ${}_{\sigma}G = {}_{t\tau}G$, then ${}_zG \simeq {}_{t\tau}G$. We have ${}_{\sigma}G(\mathbb{C}) = G(\mathbb{C})$, but the complex conjugation in ${}_{\sigma}G(\mathbb{C})$ is given by

$${}^*\bar{g} = \sigma(\bar{g}) = \text{Inn}(t)(\tau(\bar{g})).$$

Note that $\text{Inn}(t)$ acts trivially on ${}_{\tau}T$, hence also on ${}_{\tau}W$, because ${}_{\tau}W \subset \text{Aut}({}_{\tau}T)$. We see that ${}_{t\tau}T = {}_{\tau}T$ and ${}_{t\tau}W = {}_{\tau}W$.

We consider the group $W_0 := W_0({}_{t\tau}G) = W_0({}_{\tau}G)$; see Section 1. We have $W_0(\mathbb{C}) = W_0(\mathbb{R}) = {}_{\tau}W(\mathbb{R})$; see [Bo14, Section 7]. Clearly ${}_{\tau}W(\mathbb{R}) = W^{\tau}(\mathbb{C})$, hence $W_0(\mathbb{R}) = W^{\tau}(\mathbb{C})$. The group $W_0(\mathbb{R})$ acts on $H^1(\mathbb{R}, {}_{t\tau}T) = H^1(\mathbb{R}, {}_{\tau}T)$ as in formula (2), and it acts on the set of labelings $L(D^{\tau})$ via (21). We wish to describe this action explicitly.

Note that if D is of type \mathbf{A}_{2n} , then $D^{\tau} = \emptyset$, $T(D^{\tau}) = 1$, $H^1(\mathbb{R}, {}_{\tau}T) = 1$, $H^1(\mathbb{R}, {}_{\tau}G) = 1$ (in this case ${}_{\tau}G \simeq \mathbf{SL}_{2n+1}$). From now till the end of this section we shall assume that D is not of type \mathbf{A}_{2n} . Then from the classification of Dynkin diagrams we know that for any $j \in D \setminus D^{\tau}$, the vertices j and $\tau(j)$ are not connected by an edge, and therefore, the reflections r_j and $r_{\tau(j)}$ commute.

Lemma 4.1. *Assume that D is not of type \mathbf{A}_{2n} . Then the group $W_0(\mathbb{R})$ is generated by the reflections r_i for $i \in D^{\tau}$ and by the products $r_j \cdot r_{\tau(j)}$ for $j \in D \setminus D^{\tau}$.*

Proof. We have

$$W_0(\mathbb{R}) = {}_{\tau}W(\mathbb{R}) = W^{\tau}(\mathbb{C}).$$

Now the lemma follows from [Ca72, Proposition 13.1.2]. \square

Lemma 4.2. *Assume that D is not of type \mathbf{A}_{2n} . Let $j \in D \setminus D^{\tau}$. Then the product $r_j \cdot r_{\tau(j)}$ acts trivially on $H^1(\mathbb{R}, {}_{\sigma}T) = H^1(\mathbb{R}, {}_{\tau}T)$, where $\sigma = \text{Inn}(t) \circ \tau$.*

Proof. Let $b \in Z^1(\mathbb{R}, {}_{\tau}T)$. By diagram (21) we may assume that $b \in T(D^{\tau})(\mathbb{R})_2 \subset T(\mathbb{C})_2$. Let $\mathbf{b} = (b_k) \in (\mathbb{Z}/2\mathbb{Z})^D$ be the corresponding labeling of D such that $b = \prod_k (\alpha_k^{\vee}(-1))^{b_k}$.

We set $w_{j,\tau(j)} = r_j r_{\tau(j)}$. Let $G_j = G_{\alpha_j}$ denote the simple 3-dimensional subgroup of G corresponding to the simple root α_j . Choose a representative $n_j \in G_j(\mathbb{R}) \cap \mathcal{N}_G(T)(\mathbb{R})$ of $r_j \in W(\mathbb{R})$. Set $n_{\tau(j)} = \tau(n_j) \in G_{\tau(j)}(\mathbb{R})$. Since the vertices j and $\tau(j)$ are not connected by an edge, the subgroups G_j and $G_{\tau(j)}$ of G commute, hence n_j and $n_{\tau(j)}$ commute. Set $n_{j,\tau(j)} := n_j n_{\tau(j)}$, then $n_{j,\tau(j)} \in \mathcal{N}_G(T)(\mathbb{R})^{\tau}$ and $n_{j,\tau(j)}$ represents $w_{j,\tau(j)}$.

We consider the action (2) of $w_{j,\tau(j)}$ on $H^1(\mathbb{R}, {}_{t\tau}T)$. We write w for $w_{j,\tau(j)}$ and n for $n_{j,\tau(j)}$, then we have

$$w * [b] := [nbn^{-1} \cdot n\tilde{t}\tau(\bar{n})^{-1}\tilde{t}^{-1}] = [nbn^{-1} \cdot n\tilde{t}n^{-1}\tilde{t}^{-1}],$$

because $\tau(\bar{n}) = n$. Thus the action (2) of $w \in W_0(\mathbb{R})$ on $Z^1(\mathbb{R}, {}_{t\tau}T)$ is compatible with the action (8) of $w \in W(\mathbb{C})$ on $T(\mathbb{C})_2$.

We consider the action (8) of $W(\mathbb{C})$ on $T(\mathbb{C})_2$. Then Lemma 3.1 is applicable, and it implies that the move \mathcal{M}_j corresponding to the reflection $r_j \in W(\mathbb{C})$ can change only the j -coordinate b_j of \mathbf{b} . Now consider $w_{j,\tau(j)} = r_{\tau(j)}r_j \in W_0(\mathbb{R}) \subset W(\mathbb{C})$ for $j \in D \setminus D^{\tau}$, then we see that $\mathcal{M}_{\tau(j)}\mathcal{M}_j$ can change only the j - and the $\tau(j)$ -coordinates of \mathbf{b} . In particular, if we write $\mathbf{b}' = (\mathcal{M}_j\mathcal{M}_{\tau(j)})\mathbf{b}$, then $b'_i = b_i$ for any $i \in D^{\tau}$.

Since $w_{j,\tau(j)} \in W_0(\mathbb{R})$, we see that $\mathbf{b}' := w_{j,\tau(j)}(\mathbf{b})$ is contained in $Z^1(\mathbb{R}, {}_{\sigma}T)$. Since $b'_i = b_i$ for any $i \in D^{\tau}$, by formula (21) $\mathbf{b}' \sim \mathbf{b}$ in $Z^1(\mathbb{R}, {}_{\sigma}T)$. Thus $w_{j,\tau(j)} = r_{\tau(j)}r_j$ acts trivially on $H^1(\mathbb{R}, {}_{\sigma}T)$. \square

Lemma 4.3. *Let $a \in T(D^{\tau})(\mathbb{R})_2 \subset Z^1(\mathbb{R}, {}_{\sigma}T)$. Let $i \in D^{\tau}$, and write $[a'] = r_i[a]$, where $a' \in T(D^{\tau})(\mathbb{R})_2 \subset Z^1(\mathbb{R}, {}_{\sigma}T)$, and $[a] \mapsto r_i[a]$ refers to the action (2) of $r_i \in W_0(\mathbb{R})$ on*

$H^1(\mathbb{R}, {}_\sigma T)$. Write

$$a = \prod_{j \in D^\tau} (\alpha_j^\vee(-1))^{a_j}, \quad a' = \prod_{j \in D^\tau} (\alpha_j^\vee(-1))^{a'_j}$$

Then $a'_j = a_j$ for $j \neq i$ and

$$(22) \quad a'_i = a_i + t_i + \sum'_{k \succeq i, k \in D^\tau} a_k,$$

where $(-1)^{t_i} = \alpha_i(t)$ and the sum is taken over the neighbors k of i lying in D^τ .

Proof. Write $a = \prod_{j \in D} (\alpha_j^\vee(-1))^{a_j}$, then $a_j = 0$ for $j \in D \setminus D^\tau$. Now let $i \in D^\tau$, then arguing as in the proof of Lemma 4.2, we see that the action (2) of $r_i \in W_0(\mathbb{R})$ is compatible with the action (8), where $n = n_i \in G_i(\mathbb{R}) \cap \mathcal{N}_G(T)(\mathbb{R})$. By Lemma 3.1 this action is given by formula (12), i.e., by formula (22), where the sum is taken over *all* neighbors k of i in D . However, if $k \in D \setminus D^\tau$, then $a_k = 0$. We see that in formula (22) we may take the sum only over neighbors k of i contained in D^τ , as required. \square

The following theorem was announced in [Bo88]. It reduces computing the Galois cohomology of an outer form of a simply connected compact group G to computing the Galois cohomology of an inner form of some other simply connected compact group (of type \mathbf{A}_l for some l).

Theorem 4.4 ([Bo88, Theorem 3]). *Let G be a simply connected, simple, compact linear algebraic group over \mathbb{R} . Let T, R, Π and D be as in Section 2. Let τ be an automorphism of order 2 of the Dynkin diagram D of $G_{\mathbb{C}}$ (then D is simply laced). Let $l = \#D^\tau$. Let $G(D^\tau) \subset G$ be the \mathbb{R} -subgroup of type \mathbf{A}_l corresponding to the Dynkin subdiagram D^τ of D . Let $t \in T^{\text{ad}}(D^\tau)(\mathbb{R})_2 \subset {}_\tau T^{\text{ad}}(\mathbb{R})_2$. Then the natural embedding ${}_t G(D^\tau) \hookrightarrow {}_t \tau G$, obtained by twisting by t from the embedding $G(D^\tau) \hookrightarrow {}_\tau G$, induces a bijection*

$$H^1(\mathbb{R}, {}_t G(D^\tau)) \xrightarrow{\sim} H^1(\mathbb{R}, {}_t \tau G).$$

Proof. The embedding $T(D^\tau) \hookrightarrow {}_\tau T$ induces an isomorphism

$$(23) \quad H^1(\mathbb{R}, T(D^\tau)) \xrightarrow{\sim} H^1(\mathbb{R}, {}_\tau T).$$

The group $W(D^\tau)(\mathbb{R})$ acts on the left-hand side of (23); this group is generated by r_i for $i \in D^\tau$. The group $W_0(\mathbb{R})$ acts on the right-hand side; by Lemma 4.1 this group is generated by r_i for $i \in D^\tau$ and by $r_{\tau(j)} r_j$ for $j \in D \setminus D^\tau$. By Lemma 4.2 the products $r_{\tau(j)} r_j$ for $j \in D \setminus D^\tau$ act trivially on the right-hand side of (23). Comparing formulas (12) and (22), we see that the actions of a reflection r_i for $i \in D^\tau$ on the left-hand side and the right-hand side of (23) are compatible. Thus we obtain a bijection of the quotients:

$$H^1(\mathbb{R}, {}_t G(D^\tau)) = W(D^\tau)(\mathbb{R}) \backslash T(D^\tau)(\mathbb{R})_2 \xrightarrow{\sim} W_0(\mathbb{R}) \backslash H^1(\mathbb{R}, {}_\tau T) = H^1(\mathbb{R}, {}_t \tau G),$$

where the left-hand and right-hand equalities are bijections of Proposition 1.2. \square

For $i \in D^\tau$ we define the move \mathcal{M}_i by $\mathcal{M}_i \mathbf{a} = \mathbf{a}'$, where $\mathbf{a} = (a_j)_{j \in D^\tau}$ and $\mathbf{a}' = (a'_j)_{j \in D^\tau}$, and a_j, a'_j are as in Lemma 4.3. We may take $t \in T^{\text{ad}}(D^\tau)(\mathbb{R})_2 \subset {}_\tau T^{\text{ad}}(\mathbb{R})_2$ from [OV90, Table 7, Type III]. Our t corresponds to the black vertex of the Kac diagram (a Kac diagram of type III has exactly one black vertex). If the black vertex has number 0, then $t = 1$, so ${}_t \tau G = {}_\tau G$, hence $t_i = 0$ for all i , and therefore, \mathcal{M}_i is given by formula (5), where the sum is taken over the neighbors k of i lying in D^τ . If the black vertex has number $i \neq 0$, then $i \in D^\tau$, and we take $t = u_i$, i.e., $t_j = \delta_{ij}$. Thus \mathcal{M}_j for $j \neq i$ is given by formula

$$(24) \quad a'_j = a_j + \sum'_{k \succeq j, k \in D^\tau} a_k,$$

while \mathcal{M}_i is given by formula (19),

$$(25) \quad a'_i = a_i + 1 + \sum'_{k \succeq i, k \in D^\tau} a_k,$$

where again in both (24) and (25) the sum is taken over the neighbors k lying in D^τ . We obtain a coloring \mathbf{t} of D^τ , which is trivial if the black vertex of the Kac diagram has number 0.

From diagram (21) and Theorem 4.4 we obtain a commutative diagram

$$(26) \quad \begin{array}{ccccccc} L(D^\tau) & \xrightarrow{\sim} & T(D^\tau)(\mathbb{R})_2 & \xrightarrow{\sim} & H^1(\mathbb{R}, T(D^\tau)) & \longrightarrow & H^1(\mathbb{R}, {}_tG(D^\tau)) \\ \left(\uparrow \right) & & \left(\uparrow \right) & & \downarrow \sim & & \downarrow \sim \\ L(D)^\tau & \xrightarrow{\sim} & {}_\tau T(\mathbb{R})_2 & \longrightarrow & H^1(\mathbb{R}, {}_\tau T) & \longrightarrow & H^1(\mathbb{R}, {}_{t\tau}G) \end{array}$$

The map $L(D)^\tau \rightarrow H^1(\mathbb{R}, {}_{t\tau}G)$ of the bottom row of this diagram is surjective by Lemma 1.1 and Proposition 1.2. The next corollary is clear from diagram (26).

Corollary 4.5. *Two labelings $\mathbf{a}, \mathbf{a}' \in L(D)^\tau$ have the same image in $H^1(\mathbb{R}, {}_{t\tau}G)$ if and only if their images in $L(D^\tau)$ (i.e., their restrictions to D^τ) lie in the same equivalence class in $L(D^\tau, \mathbf{t})$. In particular, a labeling $\mathbf{a} \in L(D)^\tau$ maps to $[1]$ if and only if its restriction to D^τ lies in the equivalence class of 0 in $L(D^\tau, \mathbf{t})$.*

We say that two labelings $\mathbf{a}, \mathbf{a}' \in L(D)^\tau$ are *equivalent* if their restrictions to D^τ lie in the same equivalence class in $L(D^\tau, \mathbf{t})$. We denote by $\text{Cl}(D, \tau, \mathbf{t})$ the set of equivalence classes in $L(D)^\tau$ with respect to τ and \mathbf{t} , then we have a bijection

$$(27) \quad \text{Cl}(D, \tau, \mathbf{t}) \xrightarrow{\sim} H^1(\mathbb{R}, {}_{t\tau}G).$$

The restriction map $L(D)^\tau \rightarrow L(D^\tau)$ induces a bijection

$$(28) \quad \text{Cl}(D, \tau, \mathbf{t}) \xrightarrow{\sim} \text{Orb}(D^\tau, \mathbf{t})$$

In the case when ${}_zG$ is an *inner* form of the compact group G , we have $\tau = \psi(z) = 1 \in \text{Aut}(D)$, and we set $\text{Cl}(D, \tau, \mathbf{t}) = \text{Orb}(D, \mathbf{t})$ in this case. Then we have bijection (27) also when $\tau = 1$.

5. ORBITS: DEFINITIONS AND TERMINOLOGY

Starting with the next section, we solve the Reeder puzzles case by case, i.e., describe the sets of equivalence classes $\text{Cl}(D, \tau, \mathbf{t})$. Corollary 4.5 reduces the case of an outer form of a compact group to the case of an inner form of another compact group. In the case of an inner form we determine the set $\text{Orb}(D, \mathbf{t})$ of the orbits of the group W generated by the moves \mathcal{M}_i (i.e., reflections r_{α_i}) acting on the set $L(\mathbf{t}D)$. Here $L(\mathbf{t}D)$ is the set of labelings $\mathbf{a} = (a_1, \dots, a_n)$ corresponding to a twisting diagram $\mathbf{t}D$ with vertices $i = 1, \dots, n$, where $a_i \in \mathbb{Z}/2\mathbb{Z}$ and each i corresponds to the simple root α_i . We number the vertices of D as in Onishchik and Vinberg [OV90, Table 1].

By (*connected*) *components* of a labeling of a Dynkin diagram we mean the connected components of the graph obtained by removing the vertices with zeros and the corresponding edges. For example, the following labeling of \mathbf{A}_9 has 3 connected components:

$$1-1-0-1-0-0-1-1-1 \quad \mapsto \quad 1-1 \quad 1 \quad 1-1-1.$$

For some diagrams D the number of components of a labeling is an invariant of the action of W . For some others, the parity of the number of components is an invariant.

By a *fixed labeling* we mean a fixed point of the action of W , that is, a labeling which is fixed under all moves \mathcal{M}_i . For example, for the action of Lemma 2.1 on \mathbf{A}_5 , the labelings

$$0-0-0-0-0 \quad \text{and} \quad 1-0-1-0-1$$

are fixed.

We say that two vertices i, j of a Dynkin diagram D are neighbors if they are connected by an edge (single or multiple). We say that i is a vertex of degree d if it has exactly d neighbors. We are especially interested in vertices of degree 3. The Dynkin diagrams \mathbf{D}_n ($n \geq 4$), \mathbf{E}_6 , \mathbf{E}_7 and \mathbf{E}_8 have vertices of degree 3. Now let D be a Dynkin diagram with a vertex i of degree 3, and let \mathbf{a} be a labeling of D that looks near i like

$$(29) \quad \dots - 1 - \underset{\substack{| \\ 1}}{1} - 1 - \dots$$

The move \mathcal{M}_i of Lemma 2.1 splits the component of i to three components (because D has no cycles):

$$(30) \quad \dots - 1 - \underset{\substack{| \\ 1}}{0} - 1 - \dots$$

and therefore increases the number of components by 2. We call this process *splitting at i* . The reverse process is called *unsplitting*.

Let G be a group with twisting diagram ${}_t D$ and the set of labelings $L({}_t D)$. We denote by $\text{Orb}({}_t D)$ the set of orbits in $L({}_t D)$ under the action of the Weyl group, i.e., the set of equivalence classes of labelings with respect to the moves. We denote the number of orbits by $\#\text{Orb}({}_t D)$.

In Sections 6 – 14 below we describe the pointed sets $\text{Cl}(D, \tau, t)$ for simply connected groups of types $\mathbf{A}_n - \mathbf{G}_2$. Since these sections may be regarded as parts of a table, most proofs are omitted. In Section 6 we introduce notation which will be used in subsequent sections.

6. GROUPS OF TYPE \mathbf{A}_n

6.1. The compact group $\text{SU}(n+1)$ of type $\mathbf{A}_n^{(0)}$. Here $n \geq 1$. The Dynkin diagram is

$$\overset{1}{\circ} - \dots - \overset{n}{\circ}$$

The Weyl group acts by the moves that are described in Lemma 2.1. We denote the compact form of the complex group of type \mathbf{A}_n by $\mathbf{A}_n^{(0)}$. The superscript 0 shows that the group is compact and the diagram is uncolored.

Lemma 6.1. *For $\mathbf{A}_n^{(0)}$:*

- (a) *Moves do not change the number of components.*
- (b) *Every component can be reduced to length 1, e.g.*

$$0 - 1 - 1 - 1 - 0 \quad \mapsto \quad 0 - 0 - 1 - 0 - 0 .$$

- (c) *Components may be pushed so that the space between components is of length 1, e.g.*

$$1 - 0 - 0 - 1 - 0 \quad \mapsto \quad 1 - 0 - 1 - 0 - 0 .$$

Notation 6.2. By ξ_r^n (or just ξ_r) we mean the labeling of $\mathbf{A}_n^{(0)}$ of the form

$$\xi_r = 1 - 0 - \overset{1-0}{\dots} - \underset{r}{1} - 0 - \dots$$

which has r components packed maximally to the left, namely,

$$(\xi_r^n)_i = \begin{cases} 1 & \text{if } i = 1, 3, \dots, 2r - 1 , \\ 0 & \text{otherwise .} \end{cases}$$

By η_r^n (or just η_r) we mean the labeling of $\mathbf{A}_n^{(0)}$ which has r components packed maximally to the right, namely

$$(\eta_r^n)_i = \begin{cases} 1 & \text{if } i = n, n-2, \dots, n-2(r-1), \\ 0 & \text{otherwise.} \end{cases}$$

Example 6.3. $\xi_3^7 = 1-0-1-0-1-0-0$, $\eta_2^7 = 0-0-0-0-1-0-1$.

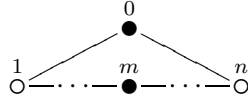
Lemma 6.4. Any labeling of $\mathbf{A}_n^{(0)}$ with r components is equivalent to ξ_r^n and to η_r^n .

Thus, in $\mathbf{A}_n^{(0)}$ the number of components is an invariant which fully characterizes orbits.

Corollary 6.5. The orbit of zero in $L(\mathbf{A}_n^{(0)})$ consists of one labeling ξ_0 . As *representatives of orbits* we can take $\xi_0, \xi_1, \dots, \xi_r$, where $r = \lceil n/2 \rceil$. We have

$$(31) \quad \#\text{Orb}(\mathbf{A}_n^{(0)}) = r + 1 = \lceil n/2 \rceil + 1 = \begin{cases} k + 1 & \text{if } n = 2k, \\ k + 2 & \text{if } n = 2k + 1. \end{cases}$$

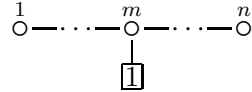
6.2. The group $\text{SU}(m, n+1-m)$ ($1 \leq m \leq \lceil n/2 \rceil$) with twisting diagram $\mathbf{A}_n^{(m)}$. The group G is the special unitary group $\text{SU}(m, n+1-m)$ of the diagonal Hermitian form with m times -1 and $n+1-m$ times $+1$ on the diagonal. Here we assume that $1 \leq m \leq \lceil n/2 \rceil$, but the results remain valid for all $1 \leq m \leq n$. The Kac diagram of G is



see [OV90, Table 7]. We obtain the *twisting diagram* $\mathbf{A}_n^{(m)}$ by removing vertex 0 from the Kac diagram:



The superscript in $\mathbf{A}_n^{(m)}$ refers to the twisting (black) vertex m in the twisting diagram, with respect to the numbering of Onishchik and Vinberg [OV90]. We construct the augmented diagram



as in Construction 3.4, i.e. by formally adding a new vertex with constant label 1, called boxed 1, connected to the twisting vertex by a simple edge. This retains the set of labelings and the set of orbits.

Notation 6.6. Let $\mathbf{a} = (a_i) \in L(\mathbf{A}_n^{(m)})$. We have a schematic diagram:

$$(32) \quad \text{LHS} - a_m - \text{RHS}$$

where LHS denotes the left-hand side and RHS denotes the right-hand side. We denote by $l(\mathbf{a})$ the number of components of \mathbf{a} in LHS (to the left of the twisting vertex m), and by $r(\mathbf{a})$ the number of components of \mathbf{a} in RHS (to the right of the twisting vertex m), in both cases not taking into account the component of the boxed 1.

Remark 6.7. For $\mathbf{A}_n^{(m)}$:

- (i) Any labeling \mathbf{a} is equivalent to a labeling \mathbf{a}' with $a'_m = 0$.
- (ii) For the schematic diagram (32), if $l(\mathbf{a}) \geq 1$ and $r(\mathbf{a}) \geq 1$, then the rightmost component in LHS and the leftmost component in RHS can be made to cancel each other out by unsplitting at vertex m (which is a vertex of degree 3 for $1 < m < n$). In other words, if $l(\mathbf{a}), r(\mathbf{a}) \geq 1$, then \mathbf{a} is equivalent to some labeling \mathbf{a}' with $l(\mathbf{a}') = l(\mathbf{a}) - 1$ and $r(\mathbf{a}') = r(\mathbf{a}) - 1$.

- (iii) A component cannot pass from one hand side to the opposite hand side. In other words, if $\mathbf{a} \sim \mathbf{a}'$, then $r(\mathbf{a}) - l(\mathbf{a}) = r(\mathbf{a}') - l(\mathbf{a}')$.

Proposition 6.8. *Two labelings $\mathbf{a}, \mathbf{a}' \in L(\mathbf{A}_n^{(m)})$ are equivalent if and only if $r(\mathbf{a}) - l(\mathbf{a}) = r(\mathbf{a}') - l(\mathbf{a}')$, and this invariant $r(\mathbf{a}) - l(\mathbf{a})$ can take values between $-\lceil(m-1)/2\rceil$ and $\lceil(n-m)/2\rceil$.*

Notation 6.9. Let p, q be integers, $0 \leq p \leq \lceil(m-1)/2\rceil$ and $0 \leq q \leq \lceil(n-m)/2\rceil$. We write

$$(33) \quad (p|q) := \eta_p^{m-1} - \underset{\boxed{1}}{0} - \xi_q^{n-m}$$

We have $l(p|q) = p$ and $r(p|q) = q$.

Corollary 6.10. *For $\mathbf{A}_n^{(m)}$:*

- (i) **The orbit of zero** is the set of the labelings \mathbf{a} such that $l(\mathbf{a}) = r(\mathbf{a})$.
(ii) As **representatives of orbits** we can take

$$(0|0) = \eta_0 - \underset{\boxed{1}}{0} - \xi_0, \quad (p|0) = \eta_p - \underset{\boxed{1}}{0} - \xi_0, \quad (0|q) = \eta_0 - \underset{\boxed{1}}{0} - \xi_q$$

with $1 \leq p \leq \lceil(m-1)/2\rceil$ and $1 \leq q \leq \lceil(n-m)/2\rceil$.

- (iii) **The number of orbits** is

$$(34) \quad \begin{aligned} \#\text{Orb}(\mathbf{A}_n^{(m)}) &= \lceil(m-1)/2\rceil + 1 + \lceil(n-m)/2\rceil \\ &= \begin{cases} k+1 & \text{if } n = 2k, \\ k+1 & \text{if } n = 2k+1 \text{ and } m \text{ is odd,} \\ k+2 & \text{if } n = 2k+1 \text{ and } m \text{ is even.} \end{cases} \end{aligned}$$

6.3. Outer forms of $\mathbf{SU}(n+1)$. Here τ is the nontrivial involutive automorphism of the Dynkin diagram $D = \mathbf{A}_n$, where $n \geq 2$.

Case $G = \mathbf{SL}(n+1)$, $n = 2k$. Then $D^\tau = \emptyset$, and it is well known that $H^1(\mathbb{R}, \mathbf{SL}(n+1)) = 1$.

Case $G = \mathbf{SL}(n+1)$, $n = 2k+1$. Then $\#D^\tau = 1$, $D^\tau = \bullet$ and again it is well known that $H^1(\mathbb{R}, \mathbf{SL}(n+1)) = 1$.

Case $G = \mathbf{SL}(k+1, \mathbb{H})$, where \mathbb{H} denotes the Hamilton quaternions. Then $n = 2k+1$, $\#D^\tau = 1$, $D^\tau = \circ$, $\#\text{Orb}(\mathbf{A}_1) = 2$. The orbit of zero in $L(\mathbf{A}_1)$ consists of 0. **The class of zero** in $L(D)^\tau$ consists of the labelings whose restriction to D^τ is 0, namely with $a_{k+1} = 0$. The other class consists of the labelings with $a_{k+1} = 1$.

7. GROUPS OF TYPE \mathbf{B}_n

7.1. The compact group $\mathbf{Spin}(2n+1)$ of type $\mathbf{B}_n^{(0)}$. The Dynkin diagram is

$$\overset{1}{\circ} - \dots - \overset{n-1}{\circ} \Rightarrow \overset{n}{\circ},$$

where $n \geq 2$.

We write a labeling $\mathbf{b} \in L(\mathbf{B}_n^{(0)})$ as $\mathbf{b} = (\mathbf{a} \Rightarrow \varkappa)$, where $\mathbf{a} \in L(\mathbf{A}_{n-1}^{(0)})$ and $\varkappa \in \{0, 1\}$.

Note that the labeling

$$\ell_1^{(0)} = \xi_0 \Rightarrow 1 = 0 - \dots - 0 \Rightarrow 1$$

is a fixed labeling. We denote by $[\ell_1^{(0)}] \in \text{Orb}(\mathbf{B}_n^{(0)})$ the orbit $\{\ell_1^{(0)}\}$ of $\ell_1^{(0)}$, and also, by slight abuse of notation, the subset $\{[\ell_1^{(0)}]\} \subset \text{Orb}(\mathbf{B}_n^{(0)})$ consisting of this orbit.

We also note that if $\mathbf{a} \in L(\mathbf{A}_{n-1}^{(0)})$, $\mathbf{a} \neq 0$, then $(\mathbf{a} \Rightarrow 1)$ is equivalent to $(\mathbf{a} \Rightarrow 0)$ in $L(\mathbf{B}_n^{(0)})$.

Corollary 7.5. *We have:*

$$\#\text{Orb}(\mathbf{B}_n^{(m)}) = \begin{cases} \#\text{Orb}(\mathbf{A}_{n-1}^{(m)}) & \text{if } m \text{ is odd,} \\ \#\text{Orb}(\mathbf{A}_{n-1}^{(m)}) + 1 & \text{if } m \text{ is even.} \end{cases}$$

Using Corollary 6.10(iii), we obtain

$$\#\text{Orb}(\mathbf{B}_n^{(m)}) = \begin{cases} k & \text{if } n = 2k \text{ and } m \text{ is odd,} \\ k + 2 & \text{if } n = 2k \text{ and } m \text{ is even,} \\ k + 1 & \text{if } n = 2k + 1 \text{ and } m \text{ is odd,} \\ k + 2 & \text{if } n = 2k + 1 \text{ and } m \text{ is even.} \end{cases}$$

7.3. The group $\text{Spin}(2n, 1)$ with twisting diagram $\mathbf{B}_n^{(n)}$. The twisting diagram and the augmented diagram are:

$$\overset{1}{\circ} - \dots - \overset{n-1}{\circ} \Rightarrow \overset{n}{\bullet} \quad \quad \overset{1}{\circ} - \dots - \overset{n-1}{\circ} \Rightarrow \overset{n}{\circ} - \boxed{1}$$

(see [OV90, Table 7] and Construction 3.4).

If $n = 2k$, we have a fixed labeling in $L(\mathbf{B}_n^{(n)})$

$$\ell_1^{(n)} = 1-0-1-\overset{0-1}{\dots}-0-1 \Rightarrow 1-\boxed{1} = \xi_k \Rightarrow 1-\boxed{1} \in L(\mathbf{B}_n^{(n)}) .$$

Proposition 7.6. *Define a map $\varphi: L(\mathbf{A}_{n-1}^{(0)}) \rightarrow L(\mathbf{B}_n^{(n)})$ by*

$$\varphi(\mathbf{a}) = (\mathbf{a} \Rightarrow 0 - \boxed{1}) ,$$

then the induced map

$$\varphi_*: \text{Orb}(\mathbf{A}_{n-1}^{(0)}) \rightarrow \text{Orb}(\mathbf{B}_n^{(n)})$$

is injective. If n is odd, then φ_ is bijective; if n is even, the image of φ_* is $\text{Orb}(\mathbf{B}_n^{(n)}) \setminus [\ell_1^{(n)}]$.*

Proposition 7.7. *The orbit of zero in $L(\mathbf{B}_n^{(n)})$ consists of two labelings:*

$$0 - \dots - 0 \Rightarrow 0 - \boxed{1} \quad \text{and} \quad 0 - \dots - 0 \Rightarrow 1 - \boxed{1} .$$

Corollary 7.8. *As representatives of orbits in $L(\mathbf{B}_n^{(n)})$ we can take*

$$\xi_0 \Rightarrow 0 - \boxed{1} , \quad \xi_1 \Rightarrow 0 - \boxed{1} , \quad \dots , \quad \xi_r \Rightarrow 0 - \boxed{1}$$

where $r = \lceil (n-1)/2 \rceil$, together with $\ell_1^{(n)}$ when n is even.

Corollary 7.9.

$$\#\text{Orb}(\mathbf{B}_n^{(n)}) = \begin{cases} \#\text{Orb}(\mathbf{A}_{n-1}^{(0)}) + 1 = k + 2 & \text{if } n = 2k , \\ \#\text{Orb}(\mathbf{A}_{n-1}^{(0)}) = k + 1 & \text{if } n = 2k + 1 . \end{cases}$$

8. GROUPS OF TYPE \mathbf{C}_n

8.1. The compact group $\text{Sp}(n)$ with diagram $\mathbf{C}_n^{(0)}$. The group G is the compact “quaternionic” group $\mathbf{Sp}(n)$ of type \mathbf{C}_n ($n \geq 3$) with Dynkin diagram

$$\overset{1}{\circ} - \dots - \overset{n-1}{\circ} \Leftarrow \overset{n}{\circ} .$$

Construction 8.1. Let $L(\mathbf{A}_{n-1}^{(0)}) \sqcup L(\mathbf{A}_{n-1}^{(n-1)})$ denote the disjoint union of the sets of labelings $L(\mathbf{A}_{n-1}^{(0)})$ and $L(\mathbf{A}_{n-1}^{(n-1)})$, which inherits an equivalence relation. We define a map

$$\varphi: L(\mathbf{A}_{n-1}^{(0)}) \sqcup L(\mathbf{A}_{n-1}^{(n-1)}) \rightarrow L(\mathbf{C}_n^{(0)})$$

sending $\mathbf{a} \in L(\mathbf{A}_{n-1}^{(0)})$ to $\mathbf{a} \Leftarrow 0$ and sending $\mathbf{a}' - \boxed{1} \in L(\mathbf{A}_{n-1}^{(n-1)})$ to $\mathbf{a}' \Leftarrow 1$. Clearly φ is a bijection.

Note that for any $\mathbf{a} \in L(\mathbf{A}_{n-1}^{(0)})$ and $\mathbf{a}' - \boxed{1} \in L(\mathbf{A}_{n-1}^{(n-1)})$, the labelings $\mathbf{a} \Leftarrow 0$ and $\mathbf{a}' \Leftarrow 1$ are not equivalent in $L(\mathbf{C}_n^{(0)})$.

Proposition 8.2. *The bijection φ of Construction 8.1 induces a bijection on orbits*

$$\varphi_*: \text{Orb}(\mathbf{A}_{n-1}^{(0)}) \sqcup \text{Orb}(\mathbf{A}_{n-1}^{(n-1)}) \xrightarrow{\sim} \text{Orb}(\mathbf{C}_n^{(0)}).$$

Corollary 8.3. *For $\mathbf{C}_n^{(0)}$:*

- (i) *The orbit of zero is just 0.*
- (ii) *As representatives for orbits we can take*

$$\xi_0 \Leftarrow 0, \quad \xi_1 \Leftarrow 0, \quad \dots, \quad \xi_r \Leftarrow 0,$$

where $r = \lceil (n-1)/2 \rceil$, and

$$\xi_0 \Leftarrow 1, \quad \xi_1 \Leftarrow 1, \quad \dots, \quad \xi_s \Leftarrow 1,$$

where $s = \lceil n/2 \rceil - 1$.

- (iii) $\#\text{Orb}(\mathbf{C}_n^{(0)}) = \#\text{Orb}(\mathbf{A}_{n-1}^{(0)}) + \#\text{Orb}(\mathbf{A}_{n-1}^{(n-1)}) = n + 1$.

Of course, it is well known that $\#H^1(\mathbb{R}, G) = n + 1$ in this case.

8.2. The diagram $\mathbf{A}_n^{(m,n)}$. (We shall need this diagram in Subsection 8.3.) Denote by $\mathbf{A}_n^{(m,n)}$ the Dynkin diagram \mathbf{A}_n with *two* black vertices m and n , where $1 \leq m < n$:

$$\overset{1}{\circ} - \dots - \overset{m-1}{\circ} - \overset{m}{\bullet} - \overset{m+1}{\circ} - \dots - \overset{n-1}{\circ} - \overset{n}{\bullet}.$$

We denote by $L(\mathbf{A}_n^{(m,n)})$ the set of labelings $\mathbf{a} = (a_i)$ of $\mathbf{A}_n^{(m,n)}$. We consider the moves \mathcal{M}_i given by formula (5) for white vertices and by formula (19) for black vertices. We construct the augmented diagram

$$\overset{1}{\circ} - \dots - \overset{m}{\circ} - \dots - \overset{n}{\circ} - \boxed{1},$$

$\boxed{1}$

by adding $\boxed{1}$ two times, and now the moves \mathcal{M}_i are given by formula (5) for all $i = 1, \dots, n$.

We consider the orbits (equivalence classes) in $L(\mathbf{A}_n^{(m,n)})$. Note that when m is odd, the labeling

$$\ell_1^{(m,n)} = 1 - 0 - \dots - \overset{1-0}{\circ} - 1 - \dots - \overset{1}{\circ} - 1 - \boxed{1}$$

$\boxed{1}$

is a fixed labeling.

Lemma 8.4. *For $\mathbf{A}_n^{(m,n)}$, we can take the following labelings as representatives of orbits: $(0|0)$, $(p|0)$ for $p = 1, \dots, \lceil (m-1)/2 \rceil$ and $(0|q)$ for $q = 1, \dots, \lceil (n-1-m)/2 \rceil$, and when m is odd, also the fixed labeling ℓ_1 .*

8.3. The group $\mathbf{Sp}(m, n-m)$ ($1 \leq m \leq \lfloor n/2 \rfloor$) with twisting diagram $\mathbf{C}_n^{(m)}$. The group G is the “quaternionic” group $\mathbf{Sp}(m, n-m)$, the unitary group of the diagonal quaternionic Hermitian form with m times -1 and $n-m$ times $+1$ on the diagonal. The twisting diagram and the augmented diagram are:

$$\overset{1}{\circ} - \dots - \overset{m}{\bullet} - \dots - \overset{n-1}{\circ} \Leftarrow \overset{n}{\circ} \qquad \overset{1}{\circ} - \dots - \overset{m}{\circ} - \dots - \overset{n-1}{\circ} \Leftarrow \overset{n}{\circ}$$

$\boxed{1}$

(see [OV90, Table 7] and Construction 3.4).

Proposition 8.5. *The bijection*

$$\varphi: L(\mathbf{A}_{n-1}^{(m)}) \sqcup L(\mathbf{A}_{n-1}^{(m,n-1)}) \rightarrow L(\mathbf{C}_n^{(m)})$$

sending $\mathbf{a} \in L(\mathbf{A}_{n-1}^{(m)})$ to $\mathbf{a} \Leftarrow 0$ and sending $\mathbf{a}' - \boxed{1} \in L(\mathbf{A}_{n-1}^{(m,n-1)})$ to $\mathbf{a}' \Leftarrow 1$, induces a bijection

$$\varphi_*: \text{Orb}(\mathbf{A}_{n-1}^{(m)}) \sqcup \text{Orb}(\mathbf{A}_{n-1}^{(m,n-1)}) \xrightarrow{\sim} \text{Orb}(\mathbf{C}_n^{(m)}).$$

Denote $(p|q < \varkappa) = \mathbf{a} \Leftarrow \varkappa$, where $\mathbf{a} = (p|q) \in L(\mathbf{A}_{n-1}^{(m)})$ and $\varkappa \in \{0, 1\}$. For example, for $\mathbf{C}_5^{(3)}$ we have

$$(1|0 < 1) = \begin{array}{ccccccc} 0 & - & 1 & - & 0 & - & 0 & - & 0 & \Leftarrow & 1 \end{array} \quad \boxed{1}$$

Corollary 8.6. *For $\mathbf{C}_n^{(m)}$:*

(i) *The orbit of zero is*

$$\{ (\mathbf{a} \Leftarrow 0) \mid \mathbf{a} \in L(\mathbf{A}_{n-1}^{(m)}), l(\mathbf{a}) = r(\mathbf{a}) \}.$$

(ii) *As representatives of orbits we can take $(p|0 < 0)$ with $p = 0, \dots, \lceil (m-1)/2 \rceil$, $(0|q < 0)$ with $q = 1, \dots, \lceil (n-1-m)/2 \rceil$, $(p|0 < 1)$ with $p = 0, \dots, \lceil (m-1)/2 \rceil$, $(0|q < 1)$ with $q = 1, \dots, \lfloor (n-1-m)/2 \rfloor = \lceil (n-2-m)/2 \rceil$, and when m is odd, the fixed labeling*

$$\ell_1^{(m,n)} = \begin{array}{ccccccccccc} 1 & - & 0 & - & \dots & - & 1 & - & \dots & - & 1 & \Leftarrow & 1 \end{array} \quad \boxed{1}$$

(iii) $\#\text{Orb}(\mathbf{C}_n^{(m)}) = \#\text{Orb}(\mathbf{A}_{n-1}^{(m)}) + \#\text{Orb}(\mathbf{A}_{n-1}^{(m,n-1)}) = n + 1$.

(Of course, it is well known that $\#H^1(\mathbb{R}, G) = n + 1$ in this case.)

8.4. The split group $\text{Sp}(2n, \mathbb{R})$ with twisting diagram $\mathbf{C}_n^{(n)}$. The twisting diagram and the augmented diagram are

$$\begin{array}{c} 1 \\ \circ \end{array} - \dots - \begin{array}{c} n-1 \\ \circ \end{array} \Leftarrow \begin{array}{c} n \\ \bullet \end{array} \qquad \begin{array}{c} 1 \\ \circ \end{array} - \dots - \begin{array}{c} n-1 \\ \circ \end{array} \Leftarrow \begin{array}{c} n \\ \circ \end{array} - \boxed{1}$$

In this case there is only one orbit, $\#\text{Orb}(\mathbf{C}_n^{(n)}) = 1$ (it is well known that $H^1(\mathbb{R}, G) = 1$ in this case).

9. GROUPS OF TYPE \mathbf{D}_n

9.1. The compact group $\text{Spin}(2n)$ of type $\mathbf{D}_n^{(0)}$. The group G is the spin group $\text{Spin}(2n)$, the universal covering of the special orthogonal group $\text{SO}(2n)$, where $n \geq 4$. The Dynkin diagram of G is

$$\begin{array}{ccccccc} 1 & & n-3 & & n-2 & & n-1 \\ \circ & - & \dots & - & \circ & - & \circ & - & \circ \\ & & & & & & | \\ & & & & & & \circ \\ & & & & & & n \end{array}$$

This diagram has a vertex of degree 3, the vertex $n-2$. For brevity we introduce the following notation: if $\mathbf{a} \in L(\mathbf{A}_{n-2}^{(0)})$, $\mathbf{a} = (a_i)$, $\varkappa, \lambda \in \{0, 1\}$, we write

$$(36) \quad \mathbf{a} \begin{array}{c} \varkappa \\ \lambda \end{array} := \begin{array}{c} \dots - a_{n-2} - \varkappa \\ | \\ \lambda \end{array}.$$

Note that for $\mathbf{D}_n^{(0)}$ the labelings

$$\ell_2^{(0)} = 0 = \xi_0^{n-2} \frac{0}{0} = 0 \dots 0 \frac{0}{0} \quad \text{and} \quad \ell_4^{(0)} = \xi_0^{n-2} \frac{1}{1} = 0 \dots 0 \frac{1}{1}$$

(a) When m is even, we have fixed labelings

$$\ell_2^{(m)} = 1-0-\overset{1-0}{\dots}-1-\underset{\boxed{1}}{0}-0-\dots-0-\underset{0}{0}$$

and

$$\ell_4^{(m)} = 1-0-\overset{1-0}{\dots}-1-\underset{\boxed{1}}{0}-0-\dots-0-\underset{1}{1}.$$

(b) When $n - m$ is even, we have fixed labelings

$$\ell_1^{(m)} = \xi_0-\underset{\boxed{1}}{0}-1-0-\overset{1-0}{\dots}-1-\underset{0}{0}-1$$

and

$$\ell_3^{(m)} = \xi_0-\underset{\boxed{1}}{0}-1-0-\overset{1-0}{\dots}-1-\underset{1}{0}-0.$$

(Cases (a) and (b) can occur together.)

Note that $[\ell_2^{(m)}]$ is in the image of the map φ_* of Theorem 9.5 below, while $[\ell_1^{(m)}], [\ell_3^{(m)}]$ and $[\ell_4^{(m)}]$ are not.

Theorem 9.5. *Consider the map $\varphi: L(\mathbf{A}_{n-2}^{(m)}) \rightarrow L(\mathbf{D}_n^{(m)})$ defined by $\mathbf{a} \mapsto \mathbf{a} \frac{0}{0}$. Then the induced map on orbits $\varphi_*: \text{Orb}(\mathbf{A}_{n-2}^{(m)}) \rightarrow \text{Orb}(\mathbf{D}_n^{(m)})$ is injective, and its image is the whole set $\text{Orb}(\mathbf{D}_n^{(m)})$ except for the fixed labelings $\ell_1^{(m)}$, $\ell_3^{(m)}$, and $\ell_4^{(m)}$ when they occur; see Remark 9.4.*

Proof. We prove the injectivity. Let $\mathbf{d} = \mathbf{a} \frac{\varkappa}{\lambda} \in L(\mathbf{D}_n^{(m)})$, where $\mathbf{a} \in L(\mathbf{A}_{n-2}^{(m)})$. Set

$$\delta(\mathbf{d}) = (\varkappa + \lambda \bmod 2)(1 - d_{n-2}) + r(\mathbf{a}) - l(\mathbf{a}),$$

where $\varkappa + \lambda \bmod 2 \in \{0, 1\} \subset \mathbb{Z}$, $d_{n-2} = a_{n-2} \in \{0, 1\} \subset \mathbb{Z}$. It is easy to check that $\delta(\mathbf{d})$ does not change by the moves in $L(\mathbf{D}_n^{(m)})$. Clearly we have $\delta(\mathbf{a} \frac{0}{0}) = r(\mathbf{a}) - l(\mathbf{a})$. Now if $\mathbf{a}, \mathbf{a}' \in L(\mathbf{A}_{n-2}^{(m)})$ and $\mathbf{a} \not\sim \mathbf{a}'$ in $L(\mathbf{A}_n^{(m)})$, then by Proposition 6.8 $r(\mathbf{a}) - l(\mathbf{a}) \neq r(\mathbf{a}') - l(\mathbf{a}')$, hence $\delta(\mathbf{a} \frac{0}{0}) \neq \delta(\mathbf{a}' \frac{0}{0})$, and therefore, $(\mathbf{a} \frac{0}{0}) \not\sim (\mathbf{a}' \frac{0}{0})$ in $L(\mathbf{D}_n^{(m)})$.

We prove the assertion about the image. There are two cases: (1) $n - m$ is odd, and (2) $n - m$ is even.

Case (1): $n - m$ is odd. Let $\mathbf{d} \in L(\mathbf{D}_n^{(m)})$. We prove that either $\mathbf{d} \sim (\dots \frac{0}{0})$ or $\mathbf{d} = \ell_4^{(m)}$. Up to equivalence, we may assume that

$$(37) \quad \mathbf{d} = \mathbf{a} \frac{\varkappa}{\lambda} = \mathbf{a}^l - \underset{\boxed{1}}{0} - \mathbf{a}^r \frac{\varkappa}{\lambda},$$

where $\mathbf{a}^l \in L(\mathbf{A}_{m-1}^{(0)})$ is the left-hand side of \mathbf{a} , $\mathbf{a}^r \in L(\mathbf{A}_{n-2-m}^{(0)})$ is the right-hand side of \mathbf{a} , and $\varkappa, \lambda \in \{0, 1\}$. If $\varkappa = \lambda = 0$, then $\mathbf{d} = \mathbf{a} \frac{0}{0}$, as required. If $\varkappa = 1, \lambda = 0$, then $\mathbf{a}^r \varkappa = \mathbf{a}^r 1 \sim (\dots 0)$ in $L(\mathbf{A}_{n-m-1}^{(0)})$, because $n - m - 1$ is even. Thus $\mathbf{d} \sim (\dots \frac{0}{0})$, as required. The case $\varkappa = 0, \lambda = 1$ is similar to the case $\varkappa = 1, \lambda = 0$.

Now assume that $\varkappa = \lambda = 1$. If $\mathbf{a}^r \neq 0$, then $\mathbf{a}^r \sim (\dots 1)$. Thus $\mathbf{d} \sim (\dots 1 \frac{1}{1}) \sim (\dots 1 \frac{0}{0})$, as required. If $\mathbf{a}^r = 0$ and either m is odd or m is even and $\mathbf{a}^l \neq \xi_{m/2}$, then we may assume that $d_{m-1} = (\mathbf{a}^l)_{m-1} = 0$. Then, applying moves, we can change d_m to 1, then change d_{m+1} to 1, ... then change d_{n-2} to 1, and finally we obtain that $\mathbf{d} \sim (\dots 1 \frac{1}{1}) \sim (\dots 1 \frac{0}{0})$, as required. If $\mathbf{a}^r = 0, m$ is even and $\mathbf{a}^l = \xi_{m/2}$, then $\mathbf{d} = \ell_4^{(m)}$, which completes the proof in Case (1).

Case (2): $n - m$ is even. Let $\mathbf{d} \in L(\mathbf{D}_n^{(m)})$. Up to equivalence, we may assume that \mathbf{d} is as in (37). If $\varkappa = \lambda = 0$, we have nothing to prove. If $\varkappa = \lambda = 1$ and $\mathbf{d} \neq \ell_4^{(m)}$, then the argument in Case (1) shows that $\mathbf{d} \sim (\dots \frac{0}{0})$, as required. Two cases remain: $\varkappa = 1$, $\lambda = 0$, and $\varkappa = 0$, $\lambda = 1$. They are similar; we treat only the case $\varkappa = 1$, $\lambda = 0$.

Consider $\mathbf{a}\varkappa = \mathbf{a}1 \in L(\mathbf{A}_{n-1}^{(m)})$. Using moves in $L(\mathbf{A}_{n-1}^{(m)})$, we can reduce $\mathbf{a}1$ to a labeling which has either 0 components right to the vertex m , or 0 components left to m . In the former case $\mathbf{d} \sim (\dots \frac{0}{0})$, as required. In the latter case, if \mathbf{d} is as in (37) and $\mathbf{a}^r 1$ has less than $k := (n - m)/2$ components, then $\mathbf{a}^r 1 \sim (\dots 0)$ and $\mathbf{d} \sim (\dots \frac{0}{0})$, as required. If $\mathbf{a}^r 1$ has k components, then $\mathbf{a}^r 1 = \xi_k$. Since $\mathbf{a}^l = 0$, we see that $\mathbf{d} = \ell_1^{(m)}$. This completes the proof in Case (2). \square

Corollary 9.6. Set $A_0 = \{\mathbf{a} \in L(\mathbf{A}_{n-2}^{(m)}) \mid l(\mathbf{a}) = r(\mathbf{a})\}$ (this is the orbit of zero in $L(\mathbf{A}_{n-2}^{(m)})$). We write $\mathbf{a} = (a_i)$. Then the orbit of zero in $L(\mathbf{D}_n^{(m)})$ is

$$\left\{ \mathbf{a} \frac{0}{0}, \mathbf{a} \frac{1}{1} \mid \mathbf{a} \in A_0 \right\} \cup \left\{ \mathbf{a} \frac{1}{0}, \mathbf{a} \frac{0}{1} \mid \mathbf{a} \in A_0, a_{n-2} = 1 \right\}.$$

Set

$$(38) \quad (p|q) \frac{\varkappa}{\lambda} := \mathbf{a} \frac{\varkappa}{\lambda} \in L(\mathbf{D}_n^{(m)}), \quad \text{where } \mathbf{a} = (p|q) \in L(\mathbf{A}_{n-2}^{(m)}), \quad \varkappa, \lambda \in \{0, 1\},$$

see formulas (33) and (36).

Corollary 9.7. For $L(\mathbf{D}_n^{(m)})$, as **representatives of orbits** we can take the labelings $(p|0) \frac{0}{0}$ for $0 \leq p \leq \lfloor m/2 \rfloor = \lceil (m-1)/2 \rceil$, the labelings $(0|q) \frac{0}{0}$ for $1 \leq q \leq \lceil ((n-2) - m)/2 \rceil$, and the fixed labelings $\ell_4^{(m)}$ and $\ell_1^{(m)}, \ell_3^{(m)}$ when they occur; see Remark 9.4.

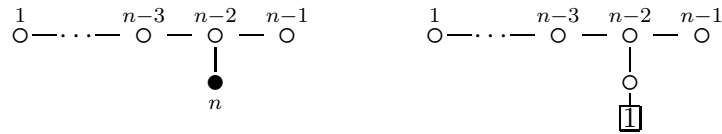
Corollary 9.8.

$$\#\text{Orb}(\mathbf{D}_n^{(m)}) = \begin{cases} k+2 & \text{if } n = 2k+1, \\ k+3 & \text{if } n = 2k \text{ and } m \text{ is even,} \\ k & \text{if } n = 2k \text{ and } m \text{ is odd.} \end{cases}$$

9.3. The group $\text{Spin}^*(2n)$ with twisting diagram $\mathbf{D}_n^{(n)}$. The group G is the “quaternionic” spin group $\text{Spin}^*(2n)$, the universal covering of $\text{SO}^*(2n)$, the special unitary group of the diagonal quaternionic skew-Hermitian form of n variables

$$ix_1 \bar{x}_1 + \dots + ix_n \bar{x}_n.$$

The twisting diagram and augmented diagram are:



(see [OV90, Table 7] and Construction 3.4).

We consider the following labelings of $\mathbf{D}_n^{(n)}$:

$$\ell_1 = \quad 0 \cdots 0 \text{---} 0 \text{---} 0 \text{---} 0 \quad \text{and} \quad \ell_2 = \quad 1 \cdots 0 \text{---} 0 \text{---} 0 \text{---} 0$$

\downarrow
 $\boxed{1}$

\downarrow
 $\boxed{1}$

Proposition 9.9. For $\mathbf{D}_n^{(n)}$ there are exactly two orbits:

1. **The orbit of zero** which consists of the labelings with odd number of components (including the boxed 1) and we can take ℓ_1 as a representative.
2. The other orbit that consists of the labelings with even number of components (including the boxed 1) and we can take ℓ_2 as a representative.

9.4. The group $\mathbf{Spin}(2m+1, 2(n-m)-1)$. Here $0 \leq m \leq \lfloor (n-1)/2 \rfloor$. The group G is an outer form of the compact group $\mathbf{Spin}(2n)$ of type \mathbf{D}_n . Here for $n > 4$ we consider the nontrivial involutive automorphism τ of the Dynkin diagram \mathbf{D}_n , while for $n = 4$ τ is a nontrivial involutive automorphism of \mathbf{D}_4 . The Kac diagram is:

$$\overset{0}{\circ} \xleftarrow{1} \overset{1}{\circ} \cdots \cdots \cdots \overset{m}{\bullet} \cdots \cdots \cdots \overset{n-2}{\circ} \xrightarrow{n-1} \overset{n-1}{\circ}$$

see [OV90, Table 7]. We erase vertex 0 and also the vertex $n-1$ coming from $D \setminus D^\tau$.

If $m = 0$, we obtain $D^\tau = \mathbf{A}_{n-2}^{(0)}$ (non-twisted). By Theorem 4.4 and (31)

$$\#\text{Orb}(D^\tau) = \lceil (n-2)/2 \rceil + 1.$$

The orbit of zero in $L(D^\tau)$ is 0. **The class of 0** in $L(D)^\tau$ consists of the labelings with zero restriction to D^τ . As **representatives of equivalence classes** we can take $\xi_0, \xi_1, \dots, \xi_r$, where $r = \lceil (n-2)/2 \rceil$. These representatives lie in $L(D^\tau)$ and hence in $L(D)^\tau$.

If $m \neq 0$, then after erasing the vertices 0 and $n-1$ we obtain the twisted diagram $\mathbf{A}_{n-2}^{(m)}$. We add boxed 1 as a neighbor to vertex m and obtain the augmented diagram

$$\begin{array}{c} \overset{1}{\circ} \cdots \cdots \cdots \overset{m}{\circ} \cdots \cdots \cdots \overset{n-2}{\circ} \\ | \\ \boxed{1} \end{array}$$

By Theorem 4.4 and formula (34)

$$\begin{aligned} \#\text{Orb}(D^\tau) &= \lceil (m-1)/2 \rceil + 1 + \lceil (n-2-m)/2 \rceil \\ &= \begin{cases} k & \text{if } n = 2k, \\ k & \text{if } n = 2k+1 \text{ and } m \text{ is odd,} \\ k+1 & \text{if } n = 2k+1 \text{ and } m \text{ is even.} \end{cases} \end{aligned}$$

The orbit of zero in $L(D, \mathbf{t})$ consists of the labelings $\mathbf{a} \in L(\mathbf{A}_{n-2}^{(m)})$ such that $l(\mathbf{a}) = r(\mathbf{a})$, see Notation 6.6. **The class of zero** in $L(D)^\tau$ consists of the labelings \mathbf{a} whose restriction $\mathbf{b} = \text{res}_{D^\tau}(\mathbf{a})$ to D^τ satisfy $l(\mathbf{b}) = r(\mathbf{b})$. As **representatives of equivalence classes** we can take $(p|0)$ where $0 \leq p \leq \lceil (m-1)/2 \rceil$, and $(0|q)$ where $1 \leq q \leq \lceil (n-2-m)/2 \rceil$. Again, these representatives lie in $L(D^\tau)$ and hence in $L(D)^\tau$.

10. GROUPS OF TYPE \mathbf{E}_6

10.1. The compact group of type $\mathbf{E}_6^{(0)}$. The Dynkin diagram of G is

$$\begin{array}{c} \overset{1}{\circ} - \overset{2}{\circ} - \overset{3}{\circ} - \overset{4}{\circ} - \overset{5}{\circ} \\ | \\ \underset{6}{\circ} \end{array}$$

Proposition 10.1. *The diagram $\mathbf{E}_6^{(0)}$ has 3 orbits. The orbits are:*

1. **The orbit of zero** consisting of 0, which is a fixed labeling.
2. The orbit consisting of all the labelings with 1 or 3 components with representative

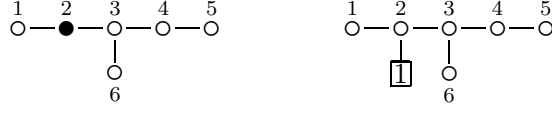
$$\ell_1 = \begin{array}{c} 1-0-0-0-0 \\ | \\ 0 \end{array}$$

3. The orbit consisting of all the labelings with 2 components with representative

$$\ell_2 = \begin{array}{c} 0-1-0-0-0 \\ | \\ 1 \end{array}$$

Remark 10.2. The moves in $L(\mathbf{E}_n)$ for $n = 6, 7, 8$ preserve the parity of the number of components.

10.2. **The group EII with twisting diagram $\mathbf{E}_6^{(2)}$.** The maximal compact subgroup is of type $\mathbf{A}_1\mathbf{A}_5$. The twisting diagram and the augmented diagram are:



(see [OV90, Table 7] and Construction 3.4).

Proposition 10.3. *The diagram $\mathbf{E}_6^{(2)}$ has 3 orbits. The orbits are:*

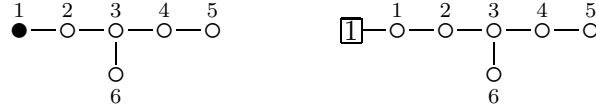
1. **The orbit of zero** consisting of all the labelings with 1 or 3 components (including the boxed 1).
2. The orbit consisting of the labelings with 2 components excluding the fixed labeling ℓ'_1 , with representative

$$\ell_3 = \begin{array}{c} 1 - 1 - 0 - 0 - 0 \\ | \quad | \\ \boxed{1} \quad 1 \end{array}$$

3. The fixed labeling

$$\ell'_1 = \begin{array}{c} 1 - 0 - 0 - 0 - 0 \\ | \quad | \\ \boxed{1} \quad 0 \end{array}$$

10.3. **The group $EIII$ of Hermitian type with twisting diagram $\mathbf{E}_6^{(1)}$.** The maximal compact subgroup of G is of type \mathbf{D}_5T^1 . The twisting diagram and the augmented diagram are:



(see [OV90, Table 7] and Construction 3.4).

Proposition 10.4. *The diagram $\mathbf{E}_6^{(1)}$ has 3 orbits. The orbits are:*

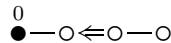
1. **The orbit of zero** consisting of the labelings with 1 or 3 components excluding the fixed labeling ℓ'_2 .
2. The orbit consisting of all the labelings with 2 components with representative

$$\ell'_3 = \begin{array}{c} \boxed{1} - 1 - 1 - 0 - 0 - 0 \\ | \\ 1 \end{array}$$

3. The fixed labeling

$$\ell'_2 = \begin{array}{c} \boxed{1} - 0 - 1 - 0 - 0 - 0 \\ | \\ 1 \end{array}$$

10.4. **The group EIV of type \mathbf{E}_6 .** This is an outer form of the compact group of type \mathbf{E}_6 with maximal compact subgroup of type \mathbf{F}_4 . The Kac diagram is



We denote by τ the nontrivial automorphism of the Dynkin diagram $D = \mathbf{E}_6$. We have $D^\tau = \overset{3}{\circ} - \overset{6}{\circ}$ and $\#\text{Orb}(D^\tau) = 2$. The orbit of zero in $L(D^\tau)$ consists of one labeling 0 of D^τ . **The equivalence class of zero** in $L(D)^\tau$ consists of the labelings whose restriction to D^τ is 0.

10.5. **The split group EI of type E_6 .** This is an outer form of the compact group of type E_6 with maximal compact subgroup of type C_4 . The Kac diagram is

$$\overset{0}{\circ} - \circ \Leftarrow \circ - \bullet$$

We have $D^\tau = \overset{3}{\circ} - \overset{6}{\bullet}$. The augmented diagram is

$$\overset{3}{\circ} - \overset{6}{\circ} - \boxed{1}$$

We have $\#\text{Orb}(D^\tau) = 2$. The orbit of zero in D^τ consists of the labelings with one component (including the boxed 1). **The equivalence class of zero** in $L(D)^\tau$ consists of the labelings \mathbf{a} such that either $a_3 = 0$ or $a_6 = 1$.

Note that $H^1(\mathbb{R}, EI)$ was earlier computed by Brian Conrad [Co14, Proof of Lemma 4.9].

11. GROUPS OF TYPE E_7

11.1. **The compact group of type $E_7^{(0)}$.** The Dynkin diagram is

$$\begin{array}{ccccccc} \overset{1}{\circ} & - & \overset{2}{\circ} & - & \overset{3}{\circ} & - & \overset{4}{\circ} & - & \overset{5}{\circ} & - & \overset{6}{\circ} \\ & & & & & & | & & & & \\ & & & & & & \overset{7}{\circ} & & & & \end{array}$$

Proposition 11.1. *The diagram $E_7^{(0)}$ has 4 orbits. The orbits are:*

1. **The orbit of zero** consisting of the fixed labeling 0.
2. The fixed labeling ℓ_3 .
3. The orbit consisting of the labelings with 1 or 3 components excluding ℓ_3 , with representative

$$\ell_1 = \begin{array}{ccccccc} 1 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 \\ & & & & & & | & & & & \\ & & & & & & 0 & & & & \end{array}$$

4. The orbit consisting of all the labelings with 2 or 4 components, with representative

$$\ell_2 = \begin{array}{ccccccc} 0 & - & 0 & - & 1 & - & 0 & - & 0 & - & 0 \\ & & & & & & | & & & & \\ & & & & & & 1 & & & & \end{array}$$

11.2. **The split group EV with twisting diagram $E_7^{(7)}$.** The maximal compact subgroup is of type A_7 . The twisting diagram and the augmented diagram are:

$$\begin{array}{ccccccc} \overset{1}{\circ} & - & \overset{2}{\circ} & - & \overset{3}{\circ} & - & \overset{4}{\circ} & - & \overset{5}{\circ} & - & \overset{6}{\circ} \\ & & & & & & | & & & & \\ & & & & & & \bullet & & & & \\ & & & & & & 7 & & & & \end{array} \qquad \begin{array}{ccccccc} \overset{1}{\circ} & - & \overset{2}{\circ} & - & \overset{3}{\circ} & - & \overset{4}{\circ} & - & \overset{5}{\circ} & - & \overset{6}{\circ} \\ & & & & & & | & & & & \\ & & & & & & \circ & & & & \\ & & & & & & | & & & & \\ & & & & & & \boxed{1} & & & & \end{array}$$

(see [OV90, Table 7] and Construction 3.4).

Proposition 11.2. *The diagram $E_7^{(7)}$ has 2 orbits. The orbits are:*

1. **The orbit of zero** is the orbit consisting of all the labelings with 1 or 3 components (including the boxed 1).
2. The orbit consisting of all the labelings with 2 or 4 components (including the boxed 1), with representative

$$m_3 = \begin{array}{ccccccc} 0 & - & 1 & - & 0 & - & 1 & - & 0 & - & 1 \\ & & & & & & | & & & & \\ & & & & & & 0 & & & & \\ & & & & & & | & & & & \\ & & & & & & \boxed{1} & & & & \end{array}$$

Note that $H^1(\mathbb{R}, EV)$ was earlier computed by Brian Conrad [Co14, Proof of Lemma 4.9].

11.3. **The group EVI with twisting diagram $\mathbf{E}_7^{(2)}$.** The maximal compact subgroup is of type $\mathbf{A}_1\mathbf{D}_6$. The twisting diagram and augmented diagram are:



(see [OV90, Table 7] and Construction 3.4).

Proposition 11.3. *The diagram $\mathbf{E}_7^{(2)}$ has 4 orbits. The orbits are:*

1. **The orbit of zero** consisting of the labelings with 1 or 3 or 5 components (including the boxed 1), excluding ℓ'_2 (see below).
2. *The fixed labeling*

$$\ell'_1 = \begin{array}{cccccc} 1 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 \\ & & \boxed{1} & & & & 0 & & & & \end{array}$$

3. *The fixed labeling*

$$\ell'_2 = \begin{array}{cccccc} 0 & - & 0 & - & 1 & - & 0 & - & 0 & - & 0 \\ & & \boxed{1} & & & & 1 & & & & \end{array}$$

4. *The orbit consisting of the labelings with 2 or 4 components (including the boxed 1) excluding ℓ'_1 , with representative*

$$\ell'_3 = \begin{array}{cccccc} 1 & - & 0 & - & 1 & - & 0 & - & 0 & - & 0 \\ & & \boxed{1} & & & & 1 & & & & \end{array}$$

Note that $H^1(\mathbb{R}, EVI)$ was earlier computed by Garibaldi and Semenov [GS10, Example 5.1] by an absolutely different method.

11.4. **The group $EVII$ of Hermitian type with twisting diagram $\mathbf{E}_7^{(1)}$.** The maximal compact subgroup is of type \mathbf{E}_6T^1 . The twisting diagram and augmented diagram are:



(see [OV90, Table 7] and Construction 3.4).

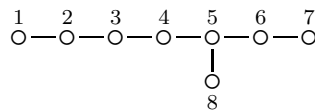
Proposition 11.4. *The diagram $\mathbf{E}_7^{(1)}$ has 2 orbits. The orbits are:*

1. **The orbit of zero** consisting of all the labelings with 1 or 3 components (including the boxed 1).
2. *The orbit consisting of all the labelings with 2 or 4 components (including the boxed 1), with representative*

$$m'_3 = \begin{array}{cccccc} \boxed{1} & - & 0 & - & 1 & - & 0 & - & 1 & - & 0 & - & 1 \\ & & & & & & & & 0 & & & & \end{array}$$

12. GROUPS OF TYPE \mathbf{E}_8

12.1. **The compact group of type $\mathbf{E}_8^{(0)}$.** The Dynkin diagram is



Proposition 12.1. *The diagram $\mathbf{E}_8^{(0)}$ has 3 orbits. The orbits are:*

1. **The orbit of zero** which contains only 0.

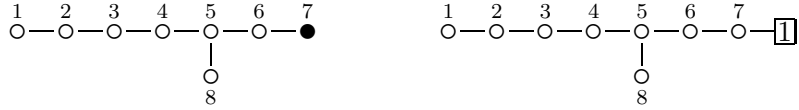
2. *The orbit consisting of all the labelings with odd number of components, with representative*

$$\ell_3 = \begin{array}{ccccccc} 0 & -1 & -0 & -1 & -0 & -0 & -0 \\ & & & & | & & \\ & & & & 1 & & \end{array}.$$

3. *The orbit consisting of all the labelings with nonzero even number of components, with representative*

$$\ell_2 = \begin{array}{ccccccc} 0 & - & 0 & - & 0 & - & 0 \\ & & & & & & | \\ & & & & & & 1 \end{array}$$

12.2. The split group $EVIII$ with twisting diagram $E_8^{(7)}$. The maximal compact subgroup is of type D_8 . The twisting diagram and the augmented diagram are:



(see [OV90, Table 7] and Construction 3.4).

Proposition 12.2. *The diagram $\mathbf{E}_8^{(7)}$ has 3 orbits. The orbits are:*

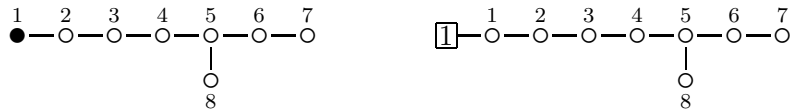
1. *The orbit of zero consisting of the labelings with odd number of components (including the boxed 1), excluding the fixed labeling ℓ'_2 .*
2. *The fixed labeling*

$$\ell'_2 = \begin{array}{ccccccc} 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 1 & - & 0 & - & \boxed{1} \\ & & & & & & | & & & & & & & & \\ & & & & & & 1 & & & & & & & & \end{array}$$

3. *The orbit consisting of all the labelings with even number of components, with representative*

$$m_3 = \begin{array}{cccccccc} 0 & -1 & -0 & -1 & -0 & -1 & -0 & \boxed{1} \\ & & & & 0 & & & \end{array}$$

12.3. **The group EIX with twisting diagram $\mathbf{E}_8^{(1)}$.** The maximal compact subgroup is of type $\mathbf{A}_1\mathbf{E}_7$. The twisting diagram and the augmented diagram are:



(see [OV90, Table 7] and Construction 3.4).

Proposition 12.3. *The diagram $\mathbf{E}_8^{(1)}$ has 3 orbits. The orbits are:*

1. **The orbit of zero** consisting of all the labelings with odd number of components (including the boxed 1).
2. The fixed labeling

$$\ell'_3 = \boxed{1}-0-1-0-1-0-0-0$$

3. The orbit consisting of the labelings with even number of components, excluding ℓ'_3 , with representative

$$m'_3 = \begin{array}{cccccccc} \boxed{1} & -0 & -1 & -0 & -1 & -0 & -1 & -0 \\ & & & & & \downarrow & & \\ & & & & & 0 & & \end{array}$$

13. GROUPS OF TYPE \mathbf{F}_4

13.1. **The compact group of type $\mathbf{F}_4^{(0)}$.** The Dynkin diagram is

$$\overset{1}{\circ} - \overset{2}{\circ} \Leftarrow \overset{3}{\circ} - \overset{4}{\circ}.$$

Proposition 13.1. *The diagram $\mathbf{F}_4^{(0)}$ has 3 orbits. The orbits are:*

1. *The orbit of zero which contains only $0-0 \Leftarrow 0-0$.*
2. *The orbit*

$$\{ 1-0 \Leftarrow 0-0, \quad 1-1 \Leftarrow 0-0, \quad \ell_1 = 0-1 \Leftarrow 0-0 \}.$$

3. *The orbit that contains the rest, with representative $\ell_2 = 1-0 \Leftarrow 1-0$.*

13.2. **The split group FI with twisting diagram $\mathbf{F}_4^{(4)}$.** The maximal compact subgroup is of type $\mathbf{C}_3\mathbf{A}_1$. The twisting diagram and the augmented diagram are:

$$\overset{1}{\circ} - \overset{2}{\circ} \Leftarrow \overset{3}{\circ} - \overset{4}{\bullet} \qquad \overset{1}{\circ} - \overset{2}{\circ} \Leftarrow \overset{3}{\circ} - \overset{4}{\circ} - \boxed{1}$$

(see [OV90, Table 7] and Construction 3.4).

Proposition 13.2. *The diagram $\mathbf{F}_4^{(4)}$ has 3 orbits. The orbits are:*

1. *The orbit of zero which consists of the labelings of the form $\mathbf{a} \Leftarrow \mathbf{a}'$, where*

$$\mathbf{a} \in L(\overset{1}{\circ} - \overset{2}{\circ}), \quad \mathbf{a}' \in L(\overset{3}{\circ} - \overset{4}{\circ} - \boxed{1}),$$

and \mathbf{a}' has only one component.

2. *The fixed labeling $\ell'_2 = 1-0 \Leftarrow 1-0 - \boxed{1}$.*
3. *The orbit*

$$\{ 0-0 \Leftarrow 1-0 - \boxed{1}, \quad 0-1 \Leftarrow 1-0 - \boxed{1}, \quad \ell_3 = 1-1 \Leftarrow 1-0 - \boxed{1} \}.$$

13.3. **The group FII with twisting diagram $\mathbf{F}_4^{(1)}$.** The maximal compact subgroup is of type \mathbf{B}_4 . The twisting diagram and the augmented diagram are:

$$\overset{1}{\bullet} - \overset{2}{\circ} \Leftarrow \overset{3}{\circ} - \overset{4}{\circ} \qquad \boxed{1} - \overset{1}{\circ} - \overset{2}{\circ} \Leftarrow \overset{3}{\circ} - \overset{4}{\circ}$$

(see [OV90, Table 7] and Construction 3.4).

Proposition 13.3. *The diagram $\mathbf{F}_4^{(1)}$ has 3 orbits. The orbits are:*

1. *The orbit of zero consisting of*

$$\{ \boxed{1}-0-0 \Leftarrow 0-0, \quad \boxed{1}-1-0 \Leftarrow 0-0, \quad \boxed{1}-1-1 \Leftarrow 0-0 \}.$$

2. *The fixed labeling $\ell'_1 = \boxed{1}-0-1 \Leftarrow 0-0$.*
3. *The orbit that contains the rest, with representative $\ell'_3 = \boxed{1}-1-1 \Leftarrow 1-0$.*

 14. GROUPS OF TYPE \mathbf{G}_2

14.1. **The compact group of type $\mathbf{G}_2^{(0)}$.** The Dynkin diagram is

$$\overset{1}{\circ} \Leftarrow \overset{2}{\circ}.$$

The description of orbits is similar to the case $\mathbf{A}_2^{(0)}$, because $3 \equiv 1 \pmod{2}$. We have $\#\text{Orb}(\mathbf{G}_2^{(0)}) = 2$. The 2 orbits are

$$\{ 0-0 \} \quad \text{and} \quad \{ 1-0, \quad 1-1, \quad 0-1 \}.$$

14.2. The split group with twisting diagram $\mathbf{G}_2^{(2)}$. The maximal compact subgroup is of type $\mathbf{A}_1\mathbf{A}_1$. The twisting diagram and the augmented diagram are:

$$\overset{1}{\circ} \leftrightsquigarrow \overset{2}{\bullet} \qquad \overset{1}{\circ} \leftrightsquigarrow \overset{2}{\circ} - \boxed{1}$$

(see [OV90, Table 7] and Construction 3.4). The description of orbits is similar to the case $\mathbf{A}_2^{(2)}$. We have $\#\text{Orb}(\mathbf{G}_2^{(2)}) = 2$. The 2 orbits are

$$\{0-0-\boxed{1}, \quad 0-1-\boxed{1}, \quad 1-1-\boxed{1}\} \quad \text{and} \quad \{1-0-\boxed{1}\}.$$

15. CONNECTED COMPONENTS IN REAL HOMOGENEOUS SPACES

Let G be a simply connected absolutely simple algebraic group over \mathbb{R} . Let $H \subset G$ be a simply connected semisimple \mathbb{R} -subgroup. Set $X = G/H$. In this section we describe our method of calculation of the number of connected components $\#\pi_0(X(\mathbb{R}))$, and give examples.

15.1. Triple (D, τ, \mathbf{t}) . Let G be a simply connected absolutely simple \mathbb{R} -group.

If G is an *outer* form of a compact group $G^{(0)}$, we can write $G = {}_t\tau G^{(0)}$ as in Section 4, where τ is an automorphism of order 2 of the Dynkin diagram D of $G_{\mathbb{C}}$. The element $t \in T^{\text{ad}}(\mathbb{R})_2$ defines a coloring \mathbf{t} of D^τ , and we may assume that the coloring comes from a Kac diagram. We obtain a triple (D, τ, \mathbf{t}) .

If G is an *inner* form of a compact group $G^{(0)}$, we can write $G = {}_tG^{(0)}$ as in Section 3, and the element $t \in T^{\text{ad}}(\mathbb{R})_2$ defines a coloring \mathbf{t} of D . In this case we set $\tau = 1$, then again \mathbf{t} is a coloring of D^τ , and again we may assume that the coloring comes from a Kac diagram. Again we obtain a triple (D, τ, \mathbf{t}) .

In both cases we have the bijection (27) $\text{Cl}(D, \tau, \mathbf{t}) \xrightarrow{\sim} H^1(\mathbb{R}, G)$.

15.2. Describing the connected components. Let H be a simply connected semisimple \mathbb{R} -subgroup of a simply connected absolutely simple \mathbb{R} -group G . We do not assume that H is simple and that H and G are inner forms of compact groups.

Let $H = H_1 \times \cdots \times H_r$ be the decomposition of H into the product of simple \mathbb{R} -groups. We may and shall assume that each H_i is absolutely simple. Let T_H be a fundamental torus of H , i.e., a maximal torus containing a maximal compact torus. Then $T_H = \prod_i T_i$ where each $T_i \subset H_i$ is a fundamental torus of H_i . We present H_i as a twisted form of a compact group as in Subsection 15.1 and obtain a triple $(D_i, \tau_i, \mathbf{t}_i)$, where D_i is the Dynkin diagram of H_i , τ_i is an automorphism of D_i with $\tau_i^2 = 1$, and \mathbf{t}_i is a coloring of $D_i^{\tau_i}$. Then we have an isomorphism $L(D_i)^{\tau_i} \xrightarrow{\sim} T_i(\mathbb{R})_2$. We set $D_H = \sqcup_i D_i$ (disjoint union), $\tau_H = \prod_i \tau_i \in \text{Aut}(D)$ (direct product of automorphisms), $L(D_H) = \bigoplus_i L(D_i)$, then $L(D_H)^{\tau_H} = \bigoplus_i L(D_i)^{\tau_i}$, and we have an isomorphism $L(D_H)^{\tau_H} \xrightarrow{\sim} T_H(\mathbb{R})_2$. We have a coloring \mathbf{t}_H of $D_H^{\tau_H}$: a vertex $v \in D_i \subset D_H$ is black in D_H if and only if it is black in D_i . We write also $L(D_H, \tau_H, \mathbf{t}_H)$ for $L(D_H)$. We define $\text{Cl}(D_H, \tau_H, \mathbf{t}_H)$ to be $\prod_i \text{Cl}(D_i, \tau_i, \mathbf{t}_i)$, then we have a bijection $\text{Cl}(D_H, \tau_H, \mathbf{t}_H) \xrightarrow{\sim} H^1(\mathbb{R}, H)$. Using results of Sections 6–14, for each i we find a set of representatives $\Xi_i \subset L(D_i, \tau_i, \mathbf{t}_i)^{\tau_i}$ of all equivalence classes in $\text{Cl}(D_i, \tau_i, \mathbf{t}_i)$. We set $\Xi = \prod_i \Xi_i \subset L(D_H, \tau_H, \mathbf{t}_H)^{\tau_H}$, then Ξ is a set of representatives of all equivalence classes in $\text{Cl}(D_H, \tau_H, \mathbf{t}_H)$, i.e., the composite map $\Xi \hookrightarrow L(D_H, \tau_H, \mathbf{t}_H)^{\tau_H} \rightarrow \text{Cl}(D_H, \tau_H, \mathbf{t}_H)$ is bijective.

Let T_G be a fundamental torus of G . We may and shall assume that $T_H \subset T_G$. We present G as a twisted form of a compact \mathbb{R} -group, then we have a triple $(D_G, \tau_G, \mathbf{t}_G)$. Using results of Sections 6–14, we compute *the class of zero* $[0]_G \subset L(D_G, \tau_G, \mathbf{t}_G)^{\tau_G}$.

The embedding $T_H(\mathbb{R})_2 \hookrightarrow T_G(\mathbb{R})_2$ induces an injective homomorphism

$$\iota: L(D_H)^{\tau_H} \rightarrow L(D_G)^{\tau_G},$$

which can be computed explicitly. Let Ξ_0 denote the preimage in Ξ of $[0]_G \subset L(D_G, \tau_G, \mathbf{t}_G)$ under the map $\Xi \hookrightarrow L(D_H, \tau_H, \mathbf{t}_H)^{\tau_H} \rightarrow L(D_G, \tau_G, \mathbf{t}_G)^{\tau_G}$, see the commutative diagram:

$$\begin{array}{ccccccc} \Xi & \longrightarrow & L(D_H, \tau_H, \mathbf{t}_H)^{\tau_H} & \longrightarrow & \mathrm{Cl}(D_H, \tau_H, \mathbf{t}_H) & \xrightarrow{\sim} & H^1(\mathbb{R}, H) \\ & & \downarrow \iota & & \downarrow & & \downarrow \\ & & L(D_G, \tau_G, \mathbf{t}_G)^{\tau_G} & \longrightarrow & \mathrm{Cl}(D_G, \tau_G, \mathbf{t}_G) & \xrightarrow{\sim} & H^1(\mathbb{R}, G) \end{array}$$

We see that Ξ_0 is in a bijection with $\ker [H^1(\mathbb{R}, H) \rightarrow H^1(\mathbb{R}, G)]$, and therefore, the cardinality of Ξ_0 answers Questions 0.2 and 0.1.

15.3. Generalities on reductive groups and Galois cohomology. Let G be a simply connected semisimple algebraic \mathbb{R} -group, $H \subset G$ be an \mathbb{R} -subgroup. The group $G(\mathbb{R})$ of \mathbb{R} -points acts on the left on $(G/H)(\mathbb{R})$.

Lemma 15.1. *Any orbit of $G(\mathbb{R})$ in $(G/H)(\mathbb{R})$ is a connected component of $(G/H)(\mathbb{R})$.*

Proof. Write $X = G/H$. Let $x \in X(\mathbb{R})$, then we have a map

$$\phi_x: G(\mathbb{R}) \rightarrow X(\mathbb{R}), \quad g \mapsto g \cdot x.$$

The differential of ϕ_x at any point $g \in G(\mathbb{R})$ is surjective, hence by the implicit function theorem the map ϕ_x is open, hence the orbits of $G(\mathbb{R})$ in $X(\mathbb{R})$ are open, hence they are open and closed. Since G is semisimple and simply connected, by [OV90, Theorem 3 in Section 5.2.1] $G(\mathbb{R})$ is connected, hence the orbits of $G(\mathbb{R})$ in $X(\mathbb{R})$ are connected, hence they are the connected components of $X(\mathbb{R})$. \square

Lemma 15.2. *Let $\varphi: S \rightarrow T$ be a homomorphism of k -tori over an algebraically closed field k (of arbitrary characteristic), and let $\varphi_*: X_*(S) \rightarrow X_*(T)$ denote the induced homomorphism of the cocharacter groups. Then*

- (i) *There is a canonical isomorphism $\mathrm{Hom}((X^*(\ker \varphi))_{\mathrm{tors}}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} (\mathrm{coker} \varphi_*)_{\mathrm{tors}}$, where by A_{tors} we denote the torsion subgroup of an abelian group A .*
- (ii) *$\#(\mathrm{coker} \varphi_*)_{\mathrm{tors}} = \#(X^*(\ker \varphi))_{\mathrm{tors}}$.*

Proof. We may assume that φ is surjective. Write $K = \ker \varphi$. From the short exact sequence

$$1 \rightarrow K \rightarrow S \xrightarrow{\varphi} T \rightarrow 1$$

we obtain an short exact sequence

$$0 \rightarrow X^*(T) \xrightarrow{\varphi^*} X^*(S) \rightarrow X^*(K) \rightarrow 0,$$

whence, by taking $\mathrm{Hom}(\cdot, \mathbb{Z})$, we obtain an exact sequence for the functor $\mathrm{Ext}_{\mathbb{Z}}$ (see e.g. [ML63, Theorem III.3.2])

$$\mathrm{Hom}(X^*(S), \mathbb{Z}) \xrightarrow{\varphi^*} \mathrm{Hom}(X^*(T), \mathbb{Z}) \rightarrow \mathrm{Ext}_{\mathbb{Z}}^1(X^*(K), \mathbb{Z}) \rightarrow \mathrm{Ext}_{\mathbb{Z}}^1(X^*(S), \mathbb{Z}) = 0.$$

where the last equality follows from the fact that $X^*(S)$ is a free abelian group. We have $\mathrm{Hom}(X^*(S), \mathbb{Z}) = X_*(S)$ and $\mathrm{Hom}(X^*(T), \mathbb{Z}) = X_*(T)$. For a finitely generated abelian group A we have $\mathrm{Ext}_{\mathbb{Z}}^1(A, \mathbb{Z}) = \mathrm{Hom}(A_{\mathrm{tors}}, \mathbb{Q}/\mathbb{Z})$, whence

$$(\mathrm{coker} \varphi_*)_{\mathrm{tors}} = \mathrm{coker} \varphi_* = \mathrm{Ext}_{\mathbb{Z}}^1(X^*(K), \mathbb{Z}) = \mathrm{Hom}(X^*(K)_{\mathrm{tors}}, \mathbb{Q}/\mathbb{Z}),$$

which proves (i), and (ii) follows immediately. \square

Corollary 15.3. *Let $\varphi: H \rightarrow G$ be a homomorphism of reductive \mathbb{C} -groups with finite kernel. Let $T_H \subset H$ and $T \subset G$ be maximal tori such that $\varphi(T_H) \subset T$. Let $\varphi_*: X_*(T_H) \rightarrow X_*(T)$ denote the induced homomorphism of the cocharacter groups. Then $\# \ker[\varphi: H \rightarrow G] = \#(\mathrm{coker} \varphi_*)_{\mathrm{tors}}$.*

Proof. Write $K = \ker \varphi = \ker[T_H \rightarrow T]$, then

$$\# \ker[\varphi: H \rightarrow G] = \#K = \#X^*(K) = \#(\operatorname{coker} \varphi_*)_{\text{tors}},$$

where the last equality follows from Lemma 15.2(ii). \square

Lemma 15.4. *Let G be a semisimple algebraic group over \mathbb{C} . Let $T \subset G$ be a maximal torus, $R = R(G, T)$ be the root system, $\Pi \subset R$ be a basis. Let $\Pi_H \subset \Pi$ be a subset, and let $R_H \subset R$ denote the subset consisting of integer linear combinations of simple roots $\alpha \in \Pi_H$ (then R_H is a root system with basis Π_H). Let H denote the algebraic subgroup of G generated by the unipotent “root” subgroups U_β for all roots $\beta \in R_H$. Then H is a semisimple group with root system R_H , and if G is simply connected, then so is H .*

Proof. The first assertion is well known, cf. [OV94, Prop. 6.1.1], so we prove the second one. Let \tilde{H} be the universal covering of H with canonical homomorphism $\varphi: \tilde{H} \rightarrow H \rightarrow G$ and maximal torus $T_{\tilde{H}}$ such that $\varphi(T_{\tilde{H}}) \subset T$. Consider the induced homomorphism

$$\varphi_*: X_*(T_{\tilde{H}}) \rightarrow X_*(T).$$

Since \tilde{H} is simply connected, the coroots $\{\alpha^\vee \mid \alpha \in \Pi_H\}$ constitute a basis of $X_*(T_{\tilde{H}})$; see [Sp98, Section 8.1.11]. It follows that

$$\operatorname{im} \varphi_* = \langle \alpha^\vee \mid \alpha \in \Pi_H \rangle \subset X_*(T).$$

Since G is simply connected, $\Pi^\vee := \{\alpha^\vee \mid \alpha \in \Pi\}$ is a basis of $X_*(T)$. It follows that $\operatorname{coker} \varphi_*$ is a free abelian group and $(\operatorname{coker} \varphi_*)_{\text{tors}} = 0$. By Corollary 15.3 we have $\# \ker \varphi = 1$, hence φ is injective and H is simply connected. \square

Lemma 15.5. *Let*

$$1 \rightarrow A \rightarrow B \xrightarrow{\psi} C \rightarrow 1$$

be a short exact sequence of algebraic \mathbb{R} -groups, where A is finite and central in B . If the order $\#A(\mathbb{C})$ of $A(\mathbb{C})$ is odd, then the induced map

$$\psi_*: H^1(\mathbb{R}, B) \rightarrow H^1(\mathbb{R}, C)$$

is bijective.

Proof. Since A is central, we have a cohomology exact sequence

$$C(\mathbb{R}) \rightarrow H^1(\mathbb{R}, A) \rightarrow H^1(\mathbb{R}, B) \xrightarrow{\psi_*} H^1(\mathbb{R}, C) \rightarrow H^2(\mathbb{R}, A);$$

see [Se94, I.5.7, Prop. 43]. Since $\#\operatorname{Gal}(\mathbb{C}/\mathbb{R}) = 2$ and $\#A(\mathbb{C})$ is odd, by [AW67, Section 6, Corollary 1 of Proposition 8] we have $H^1(\mathbb{R}, A) = 1$ and $H^2(\mathbb{R}, A) = 1$. It follows that the map ψ_* is surjective and that $\ker \psi_* = 1$. We show that any fiber of ψ_* contains only one element. Indeed, let $\beta \in H^1(\mathbb{R}, B)$ and let $b \in Z^1(\mathbb{R}, B)$ be a cocycle representing β . By [Se94, I.5.5, Cor. 2 of Prop. 39], the fiber $\psi_*^{-1}(\psi_*(\beta))$ is in a bijection with the quotient of $H^1(\mathbb{R}, A)$ by an action of the group ${}_b C(\mathbb{R})$. Since $H^1(\mathbb{R}, A) = 1$, our fiber $\psi_*^{-1}(\psi_*(\beta))$ indeed contains only one element. Thus ψ_* is bijective. \square

In Subsections 15.4–15.7 we give examples of calculations of $\#\pi_0((G/H)(\mathbb{R}))$ using results of Sections 6–14.

15.4. Example with E_7 . Let $G = EV$, the split simply connected simple \mathbb{R} -group with compact maximal torus T , of type $E_7^{(7)}$ with twisting diagram and augmented diagram

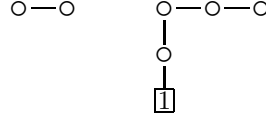


see Subsection 11.2. Let $\Pi_G = \{\alpha_1, \dots, \alpha_7\}$ be the simple roots (numbered as on the twisting diagram above). We remove vertex 3. Set $\Pi_H = \Pi_G \setminus \{\alpha_3\}$, and let H be the

corresponding semisimple \mathbb{R} -subgroup, see Lemma 15.4, with maximal torus T_H (contained in T) and with twisting diagram of type $\mathbf{A}_2^{(0)} \sqcup \mathbf{A}_4^{(1)}$



and augmented diagram:



By Lemma 15.4 the semisimple group H is simply connected. We have $H = H_1 \times H_2$, where H_1 is a compact groups of type $\mathbf{A}_2^{(0)}$ and H_2 is a twisted (noncompact) group of type $\mathbf{A}_4^{(1)}$. By Subsection 6.1, for H_1 we have $\#\text{Orb}(\mathbf{A}_2^{(0)}) = 2$ with a set of representatives

$$\Xi_1 = \{ 0-0, \quad 1-0 \}.$$

By Subsection 6.2, for H_2 we have $\#\text{Orb}(\mathbf{A}_4^{(1)}) = 3$ with a set of representatives

$$\Xi_2 = \{ \boxed{1}-0-0-0-0, \quad \boxed{1}-0-1-0-0, \quad \boxed{1}-0-1-0-1 \}.$$

We set $\Xi = \Xi_1 \times \Xi_2 \subset L(\mathbf{A}_2^{(0)} \sqcup \mathbf{A}_4^{(1)}) = L(\mathbf{A}_2^{(0)} \times L(\mathbf{A}_4^{(1)}))$, hence $\#\Xi = 2 \cdot 3 = 6$. We write down Ξ :

$$\begin{array}{ll} 0-0 & \boxed{1}-0-0-0-0 \\ 0-0 & \boxed{1}-0-1-0-0 \\ 0-0 & \boxed{1}-0-1-0-1 \\ 1-0 & \boxed{1}-0-0-0-0 \\ 1-0 & \boxed{1}-0-1-0-0 \\ 1-0 & \boxed{1}-0-1-0-1 \end{array}$$

We must compute the subset Ξ_0 of Ξ consisting of the labelings whose images in $L(\mathbf{E}_7^{(7)})$ are contained in the orbit of zero $[0]$. The homomorphism $L(\mathbf{A}_2^{(0)} \sqcup \mathbf{A}_4^{(1)}) \rightarrow L(\mathbf{E}_7^{(7)})$ is induced by the embedding $\mathbf{A}_2^{(0)} \sqcup \mathbf{A}_4^{(1)} \hookrightarrow \mathbf{E}_7^{(7)}$. By Subsection 11.2 the labelings of $\mathbf{E}_7^{(7)}$ in the orbit of zero are those with 1 or 3 components (including the boxed 1). Thus Ξ_0 consists of the following labelings of $\mathbf{A}_2^{(0)} \times \mathbf{A}_4^{(1)}$:

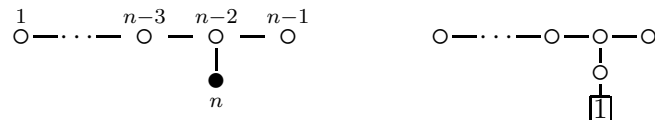
$$\begin{array}{ll} 0-0 & \boxed{1}-0-0-0-0 \\ 0-0 & \boxed{1}-0-1-0-1 \\ 1-0 & \boxed{1}-0-1-0-0 \end{array}$$

We conclude that

$$\#\pi_0((G/H)(\mathbb{R})) = \#\ker [H^1(\mathbb{R}, H) \rightarrow H^1(\mathbb{R}, G)] = \#\Xi_0 = 3.$$

Similar calculations show that if we remove vertex 2 instead of vertex 3, then $\#\pi_0((G/H)(\mathbb{R})) = 2$, and if we remove vertex 1 instead of vertex 3, then $\#\pi_0((G/H)(\mathbb{R})) = 1$, i.e. $(G/H)(\mathbb{R})$ will be connected.

15.5. Examples with $\mathbf{Spin}^*(2n)$. Let $G = \mathbf{Spin}^*(2n)$ ($n \geq 4$), the simply connected “quaternionic” \mathbb{R} -group of type $\mathbf{D}_n^{(n)}$ with twisting diagram and augmented diagram



see Subsection 9.3. Let $\Pi_G = \{\alpha_1, \dots, \alpha_n\}$ be the simple roots (numbered as on the twisting diagram above). We remove vertex $n - 1$. Set $\Pi_H = \Pi_G \setminus \{\alpha_{n-1}\}$, and let H be the corresponding semisimple \mathbb{R} -subgroup, see Lemma 15.4, with twisting diagram of type $\mathbf{A}_{n-1}^{(1)}$

$$\overset{1}{\circ} \cdots \cdots \overset{n-2}{\circ} - \bullet^n$$

and augmented diagram

$$\overset{1}{\circ} \cdots \cdots \overset{n-2}{\circ} - \overset{n}{\circ} - \boxed{1}.$$

By Lemma 15.4 the semisimple \mathbb{R} -subgroup H is simply connected. By Subsection 6.2 we can take for representatives of orbits in $L(\mathbf{A}_{n-1}^{(1)})$ the set

$$\Xi = \{\eta_i \mid 1 \leq i \leq \lceil n/2 \rceil\},$$

where η_i denotes the labeling with i components (including the boxed 1) maximally packed to the right. By Subsection 9.3, the orbit of zero in $L(\mathbf{D}_n^{(n)})$ is the set of labelings with *odd* number of components (including the boxed 1). Thus

$$\Xi_0 = \{\eta_i \mid 1 \leq i \leq \lceil n/2 \rceil, i \text{ is odd}\}.$$

We see that $\#\Xi_0$ is the number of odd numbers i between 1 and $\lceil n/2 \rceil$, i.e., $\#\Xi_0 = \lceil n/4 \rceil$. We conclude that

$$\#\pi_0((G/H)(\mathbb{R})) = \#\ker[H^1(\mathbb{R}, H) \rightarrow H^1(\mathbb{R}, G)] = \#\Xi_0 = \lceil n/4 \rceil.$$

Now, instead of removing vertex $n - 1$, let us remove vertex m with $1 \leq m \leq n - 2$:

$$\begin{array}{ccccccc} \overset{1}{\circ} & \cdots & \cdots & \overset{m-1}{\circ} & & \overset{m+1}{\circ} & \cdots & \cdots & \overset{n-3}{\circ} & - & \overset{n-2}{\circ} & - & \overset{n-1}{\circ} \\ & & & & & & & & & & \downarrow & & \\ & & & & & & & & & & \bullet & & \\ & & & & & & & & & & n & & \end{array}$$

We obtain a subgroup $H = H_1 \times H_2$, where H_1 is of type $\mathbf{A}_{m-1}^{(0)}$ (where $m - 1 = 0$ is possible) and H_2 is of type $\mathbf{D}_{n-m}^{(n-m)}$ (where $n - m = 2$ is possible). On the left of the removed vertex we can take

$$\Xi_1 = \{\xi_k \mid 0 \leq k \leq \lceil (m - 1)/2 \rceil\}$$

for representatives of orbits in $L(\mathbf{A}_{m-1}^{(0)})$, see Subsection 6.1. On the right of the removed vertex we can take

$$\Xi_2 = \{\ell_1, \ell_2\}$$

for representatives of orbits in $L(\mathbf{D}_{n-m}^{(n-m)})$ (where the labeling ℓ_1 has one component and ℓ_2 has two components, including the boxed 1), see Subsection 9.3. By Subsection 9.3 applied to G , the orbit of zero in $L(\mathbf{D}_n^{(n)})$ is the set of labelings with *odd* number of components (including the boxed 1). Now with any $\xi_k \in \Xi_1$ we associate the pair $(\xi_k, \ell) \in \Xi_1 \times \Xi_2$, where ℓ is either ℓ_1 or ℓ_2 such that the total number of components in ξ_k and ℓ is odd. We obtain a bijection $\Xi_1 \xrightarrow{\sim} \Xi_0$. Thus in this case

$$\begin{aligned} \#\pi_0((G/H)(\mathbb{R})) &= \#\ker[H^1(\mathbb{R}, H) \rightarrow H^1(\mathbb{R}, G)] \\ &= \#\Xi_0 = \#\Xi_1 = \lceil (m - 1)/2 \rceil + 1 = \lceil (m + 1)/2 \rceil. \end{aligned}$$

In particular, if $m = n - 2$, we obtain $\#\pi_0((G/H)(\mathbb{R})) = \lceil (n - 1)/2 \rceil$.

15.6. Example with $\mathbf{Spin}(2m+1, 2n+1)$. Let $G = \mathbf{Spin}(2m+1, 2n+1)$ ($m \geq 2, n \geq 3$), which is an outer form of a compact group. The Kac diagram of G is

$$\overset{0}{\circ} \xleftarrow{1} \overset{1}{\circ} \cdots \xrightarrow{m} \bullet \cdots \xrightarrow{\ell-1} \overset{\ell-1}{\circ} \xrightarrow{\ell} \overset{\ell}{\circ},$$

see Subsection 9.4 where $\ell = m + n$, see [OV90, Table 7]. We remove vertex $m + n - k$ ($2 \leq k < n$) and denote the obtained semisimple subgroup by H , then by Lemma 15.4 the group H is simply connected, and we have $H = \mathbf{SU}(m, n - k) \times \mathbf{Spin}(1, 2k + 1)$. We are interested in $\pi_0((G/H)(\mathbb{R}))$. By Theorem 4.4 applied to G and H we have a bijection $\pi_0((G'/H')(\mathbb{R})) \xrightarrow{\sim} \pi_0((G/H)(\mathbb{R}))$, where $G' = \mathbf{SU}(m, n)$ of type $\mathbf{A}_{m+n-1}^{(m)}$ and $H' = H_1 \times H_2$ with $H_1 = \mathbf{SU}(m, n - k)$ of type $\mathbf{A}_{m+n-k-1}^{(m)}$ and $H_2 = \mathbf{SU}(k)$ of type $\mathbf{A}_{k-1}^{(0)}$. Although probably one can compute $\# \pi_0((G'/H')(\mathbb{R}))$ using real algebraic geometry, we compute this number using Galois cohomology. Namely, for H_1 of type $\mathbf{A}_{m+n-k-1}^{(m)}$ we can take

$$\Xi_1 = \{((p|0), (0|q)) \mid 0 \leq p \leq \lceil (m-1)/2 \rceil, 1 \leq q \leq \lceil (n-k-1)/2 \rceil\}$$

for representatives of orbits in $L(\mathbf{A}_{m+n-k-1}^{(m)})$, see Subsection 6.2. For H_2 of type $\mathbf{A}_{k-1}^{(0)}$ we can take

$$\Xi_2 = \{\xi_i \mid 0 \leq i \leq \lceil (k-1)/2 \rceil\}$$

for representatives of orbits in $L(\mathbf{A}_{k-1}^{(0)})$, see Subsection 6.1. For G' , the orbit of zero in $L(\mathbf{A}_{m+n-1}^{(m)})$ is the set of labelings with the same number of components on the left and on the right of m , see Subsection 6.2. Thus

$$\Xi_0 = \{((p|0), \xi_p) \in \Xi_1 \times \Xi_2\}.$$

Here $0 \leq p \leq \lceil (m-1)/2 \rceil$, $0 \leq p \leq \lceil (k-1)/2 \rceil$, hence

$$\#\Xi_0 = 1 + \min(\lceil (m-1)/2 \rceil, \lceil (k-1)/2 \rceil) = \min(\lceil (m+1)/2 \rceil, \lceil (k+1)/2 \rceil).$$

We conclude that

$$\#\pi_0((G/H)(\mathbb{R})) = \#\pi_0((G'/H')(\mathbb{R})) = \#\Xi_0 = \min(\lceil (m+1)/2 \rceil, \lceil (k+1)/2 \rceil).$$

15.7. Example with \mathbf{E}_8 . Let $G = EVIII$, the split form $\mathbf{E}_8^{(7)}$ of \mathbf{E}_8 with compact maximal torus T , with Kac diagram and augmented diagram

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \bullet \\ & & & & & \downarrow & & & & & & & \\ & & & & & 8 & & & & & & & \end{array} \quad \begin{array}{cccccccc} \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \square \\ & & & & & & \downarrow & & & & & & & & \\ & & & & & & 8 & & & & & & & & \end{array}$$

see Subsection 12.2. In this example, in contrast to the two previous examples, we construct an \mathbb{R} -subgroup H of G of the same rank, and not of smaller rank. We remove vertex 4 from the Kac diagram (the extended Dynkin diagram), and do not erase vertex 0. This means that we consider the semisimple \mathbb{R} -subgroup H of G , generated (over \mathbb{C}) by the unipotent “root” subgroups U_β for all roots $\beta \in R$ that are integer linear combinations of the roots α_i , $0 \leq i \leq 8$, $i \neq 4$, where $\alpha_1, \dots, \alpha_8$ are the simple roots and α_0 is the lowest root. We obtain a maximal connected algebraic subgroup H of G [OV94, Table 5] with twisting diagram

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ \circ & - & \circ & - & \circ & - & \circ \\ & & & & \downarrow & & \\ & & & & 8 & & \end{array} \quad \begin{array}{ccc} 5 & 6 & 7 \\ \circ & - & \circ & - & \bullet \\ & & \downarrow & & \\ & & 8 & & \end{array}$$

and augmented diagram

$$(39) \quad \begin{array}{cccc} \circ & - & \circ & - & \circ & - & \circ \\ & & & & \downarrow & & \\ & & & & 8 & & \end{array} \quad \begin{array}{cccc} \circ & - & \circ & - & \circ & - & \circ & - & \square \\ & & & & \downarrow & & \\ & & & & 8 & & \end{array}$$

We compute the fundamental group $\pi_1(H_{\mathbb{C}})$ of the semisimple group H . Let \tilde{H} denote the universal covering of H . Consider the composite morphism

$$\varphi: \tilde{H} \rightarrow H \rightarrow G,$$

and let \tilde{T}_H denote the maximal torus of \tilde{H} such that $\varphi(\tilde{T}_H) = T$. We denote by $\varphi_*: X_*(\tilde{T}_H) \rightarrow X_*(T)$ the induced homomorphism of the cocharacter groups. The cocharacter group $X_*(\tilde{T}_H)$ has a basis

$$(40) \quad \alpha_0^\vee, \alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee, \widehat{\alpha_4^\vee}, \alpha_5^\vee, \alpha_6^\vee, \alpha_7^\vee, \alpha_8^\vee,$$

where $\widehat{\alpha_4^\vee}$ means that α_4^\vee is removed from the list. The cocharacter group $X_*(T)$ has a basis $\alpha_1^\vee, \dots, \alpha_8^\vee$, while the subgroup $\text{im } \varphi_* \subset X_*(T)$ is generated by the cocharacters (40). There is a linear relation between $\alpha_0^\vee, \alpha_1^\vee, \dots, \alpha_8^\vee$, in which the removed simple coroot α_4^\vee appears with coefficient 5, while α_0^\vee appears with coefficient 1; see [OV90, Table 6] or [Bou68, Planche VII, (IV)]. We see that $\text{im } \varphi_* \subset X_*(T)$ contains α_i^\vee for $i \neq 4$, and it contains $5\alpha_4^\vee$, but not α_4^\vee . Thus $\text{im } \varphi_*$ is a subgroup of index 5 in $X_*(T)$. By Corollary 15.3 the kernel of the canonical epimorphism $\tilde{H} \rightarrow H$ is of order 5, hence $\pi_1(H_{\mathbb{C}})$ is of order 5.

Since the order 5 of $\ker \varphi$ is odd, by Lemma 15.5 the induced map $H^1(\mathbb{R}, \tilde{H}) \rightarrow H^1(\mathbb{R}, H)$ is bijective, whence

$$\# \ker[H^1(\mathbb{R}, H) \rightarrow H^1(\mathbb{R}, G)] = \# \ker[H^1(\mathbb{R}, \tilde{H}) \rightarrow H^1(\mathbb{R}, G)].$$

We compute $\# \ker[H^1(\mathbb{R}, \tilde{H}) \rightarrow H^1(\mathbb{R}, G)]$. We have $\tilde{H} = \tilde{H}_1 \times \tilde{H}_2$, where \tilde{H}_1 is compact of type $\mathbf{A}_4^{(0)}$ and \tilde{H}_2 is of type $\mathbf{A}_4^{(4)}$. By Subsection 6.1 we can take

$$\Xi_1 = \{ 0-0-0-0, \quad 1-0-0-0, \quad 1-0-1-0 \}$$

as a set of representatives of orbits in $L(\mathbf{A}_4^{(0)})$. By Subsection 6.2 we can take

$$\Xi_2 = \{ 0-0-0-0-\boxed{1}, \quad 0-0-1-0-\boxed{1}, \quad 1-0-1-0-\boxed{1} \}$$

as a set of representatives of orbits in $L(\mathbf{A}_4^{(4)})$. Set $\Xi = \Xi_1 \times \Xi_2$. We denote Ξ_0 the preimage in Ξ of the orbit of zero in $L(\mathbf{E}_8^{(7)})$. By Subsection 12.2 the orbit of zero $[0] \subset L(\mathbf{E}_8^{(7)})$ consists of the labelings with odd number of components (including the boxed 1), excluding the fixed labeling ℓ'_2 . The subset of Ξ consisting of labelings with odd number of components has the following 6 labelings:

$$\begin{array}{ll} 0-0-0-0 & 0-0-0-0-\boxed{1} \\ 0-0-0-0 & 1-0-1-0-\boxed{1} \\ 0-0-0-1 & 0-0-1-0-\boxed{1} \\ 0-1-0-1 & 0-0-0-0-\boxed{1} \\ 0-1-0-1 & 1-0-1-0-\boxed{1} \\ 0-1-0-1 & 0-0-1-0-\boxed{1} \end{array}$$

and one of them ($0-0-0-0 \quad 1-0-1-0-\boxed{1}$) is the preimage of ℓ'_2 . We see that $\Xi_0 = 5$. We conclude that

$$\begin{aligned} \# \pi_0((G/H)(\mathbb{R})) &= \# \ker[H^1(\mathbb{R}, H) \rightarrow H^1(\mathbb{R}, G)] \\ &= \# \ker[H^1(\mathbb{R}, \tilde{H}) \rightarrow H^1(\mathbb{R}, G)] = \# \Xi_0 = 5. \end{aligned}$$

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